# INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES



## THE DEFINITION OF EQUIVALENCE OF COMBINATORIAL IMBEDDINGS

## ON THE STRUCTURE OF CERTAIN SEMI-GROUPS OF SPHERICAL KNOT CLASSES

## ORTHOTOPY AND SPHERICAL KNOTS

*by Barry MAZUR*

1959

UBLICATIONS MATHÉMATIQUES, N° 3

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(p. 5 à 17)

# ON THE STRUCTURE OF CERTAIN SEMI-GROUPS OF SPHERICAL KNOT CLASSES

(p. 19 à 27)

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# THE DEFINITION OF EQUIVALENCE OF COMBINATORIAL IMBEDDINGS

*By* Barry MAZUR

In treating the intriguing problem of equivalences of imbeddings of complexes K in $E^r$, a brutish technical question arises: The question of whether or not one can extend an isotopy from K to K′ to an isotopy of the entire ambient space $E^r$ onto itself (*which would bring of course,* K *to* K′).

For if one could not, a multiplicity of possible definitions of knot-equivalence would arise. And what is worse, an isotopy from K to K′ would tell little about the relationship between the respective complementary spaces.

So, the purpose of this paper is to prove just that: Any isotopy of a subcomplex $K \subseteq E^r$ to K′ can be extended to an isotopy of $E^r$ to $E^r$.

The construction of the extended isotopy is done in two stages. The first stage is to reduce the problem to one of moderately local considerations. The second is to solve the local problem remaining. The technical machinations involved in solving the local problem consists in attempting to restate it as a common extension problem.

## § 1. Terminology Section.

$E^r$ will stand for euclidean *r*-space complete with metric and linear structure. $B^r$ is to be the closed unit ball in $E^r$. By finite complex, subcomplex, and subdivision, I mean what is usually meant.

Definition. An open finite complex F is just a pair (K, L), where K and L are finite complexes, $L \subset K$ and every simplex of L is a face of a simplex of K. The *geometric* realization $|F|$ of F will be the space $|K| - |L|$, where $|X|$ is the geometric realization of the complex X.

A *simplicial map* $\varphi$ of F to F′, where F and F′ are open finite complexes:

$$F = (K, L)$$
$$F' = (K', L')$$

will be a map $\varphi$ of the simplices of K to the simplices of K′, taking L into L′, and $\partial_i \varphi(\Delta) = \varphi(\partial_i \Delta)$ as long as $\partial_i \Delta \notin L$, where $\Delta \in K$ and $\partial_i$ is the $i^{\text{th}}$ face operator. (Thus, $|\varphi|$ will be continuous as a function $|\varphi| : |F| \to |F'|$, but not necessarily on $|K|$.)

REMARK. If L is any subcomplex of a complex $K$, $K-L$ can be considered as an *open complex.*

I shall constantly confuse a complex with its geometric realization. A closed neighborhood will refer to the closure of an open set. And if $X \subseteq Y$, I define

$$\partial X = CL(X) \cap CL(E^r - X)$$

where $CL(X)$ = the closure of X in Y ; I don't refer to Y, in the terminology, because there will never be any ambiguity. If $v$ is a vertex of a complex K, then St $v$ is the complex in K generated by all simplices containing $v$. If $X \subseteq Y$, int $X = X - \partial X$. There are three words which I will have recourse to use:

1) *Simplicial map:* $\varphi : K \to K'$. This is given its usual meaning.

2) *Combinatorial map:* $\varphi : K \to K'$. This will mean simplicial with respect to some subdivision of K and K'.

3) *Piece-wise linear map:* $\varphi : K \to E^r$. This will mean linear on each simplex of K, and will be reserved for use only when the range is $E^r$.

By $1 : M \to M$, I shall mean the identity map of a set onto itself. If $A, B \subseteq E^r$, $J(A, B)$ will stand for the set $\{t\alpha + (1-t)\beta \mid \alpha \in A, \beta \in B\}$, for $0 \leqslant t \leqslant 1$.

FUNCTION SPACES. If K, L are complexes (open or not), let $M(K, L)$ be the set of all simplicial maps of K into L. Then $M(K, L)$ can be given a topology in a natural way. The easiest way to describe this topology is to imbed L in $E^N$ for large N. Then

$$M(K, L) \subseteq M(K, E^N)$$

and $M(K, E^N)$ is given a metric as follows: if $\varphi, \psi \in M(K, E^N)$

$$\delta(\varphi, \psi) = \max_{v \in V(K)} \| \varphi(v) - \psi(v) \|$$

where $V(K)$ is the set of vertices of K. $M(K, L)$ inherits a topology in this way, and it is a simple matter to show that it is independent of the imbedding $L \subseteq E^N$.

## § 2. On the definition of knot and knot-equivalence.

Whereas in the classical (one-dimensional) knot theory, a unique and natural notion of equivalence of imbedding more or less immediately presents itself, in general dimensions this is not so, and some choices must be made. For instance, by the title alone I am already committed to the genre of combinatorial as opposed to differentiable. Doubtless it is of no concern, and anyone familiar with the liaison between the two concepts can easily make the appropriate translation to the domain of differentiable imbeddings.

Then there are two possible points of view towards a complex K knotted in W.

I. The knot may be considered as the imbedding:

$$\varphi : K \to W$$

or

II. It may be looked upon as the subcomplex $K' \subseteq W$, $K' = \varphi(K)$, where the precise imbedding $\varphi : K \to K'$ has been lost.

I take the second point of view. (In most cases where I and II differ significantly, I is woefully unnatural. Also, most crucial when K is a sphere is the concept of knot addition, an operation more at home with II than I.) So, by a knotted complex K in W will be meant a subcomplex $K'$ such that there is a combinatorial homeomorphism

$$\varphi : K \to K' \qquad \text{(i.e. } K \approx K'\text{)}.$$

Finally, three notions come to mind as candidates for the definition of knot-equivalence. To distinguish them, I give them the names:

1) Isotopy equivalence.
2) Ambient isotopy equivalence.
3) Ambient homeomorphism equivalence.

My list calls for a few definitions.

## § 3. The three equivalence relations.

Definition 1. Let K and $K'$ be combinatorially isomorphic subcomplexes of $E^r$. A *continuous isotopy* $\varphi_t$ between K and $K'$ will be: a sequence of combinatorial homeomorphisms with respect to some fixed subdivision of K:

$$\varphi_t : K \to E^r$$

(alternatively referred to as

$$\varphi : I \times K \to E^r, \qquad \varphi_t(k) = \varphi(t, k)),$$

such that $\varphi_0 = 1$, and $\varphi_1 : K \overset{\approx}{\to} K'$, and $\varphi_t$ is continuous in $t$ (i.e. $\varphi_t$ considered as an arc in the function space $M(K, E^r)$). A *combinatorial isotopy* between K and $K'$ is a continuous isotopy such that if $\varphi_k : I \to E^r$ is defined to be the map $\varphi_k(t) = \varphi_t(k)$ for fixed $k \in K$, $\varphi_k$ is piece-wise linear with respect to a fixed subdivision $S(I)$, independent of $k$. I'll call K and $K'$ continuously isotopic (combinatorially isotopic) if there exists a continuous (or combinatorial) isotopy between them.

A fairly easily obtained result (mentioned to me once by M. Hirsch) simplifies things somewhat: *If* K *and* $K'$ *are continuously isotopic, they are also combinatorially isotopic.* (I omit the proof.) Therefore, in what follows, I suppress unnecessary adjectives and refer to K and $K'$ as merely: isotopic.

Definition 2. Let K, $K'$ be two subcomplexes in $E^r$. By an *ambient isotopy* between K and $K'$, I shall mean a sequence of combinatorial homeomorphisms:

$$\varphi_t : E^r \to E^r$$

such that $\varphi_t$ is (again) continuous in $t$ (in the usual function-space sense of the word). I do *not* require that $\varphi_t$ be combinatorial for fixed subdivision of $E^r$ independent of $t$. I do require, however, that $\varphi_t | K$ is an isotopy between K and K′. Finally I require $\varphi_0 = 1$. It is clear, then, that if K and K′ are ambiently isotopic, they are isotopic; every ambient isotopy restricts to an isotopy.

DEFINITION 3. If $f_t$ is the isotopy obtained by restricting the ambient isotopy $F_t$ to K, I shall say: $F_t$ covers $f_t$. And so, the first two notions of knot-equivalence are:

I) $K \underset{\iota}{\sim} K'$ if and only if K, K′ are isotopic

II) $K \underset{a}{\sim} K'$ if and only if K, K′ are ambiently isotopic.

EQUIVALENCE THEOREM. The two equivalence relations $\underset{\iota}{\sim}$ and $\underset{a}{\sim}$ are the same.

The equivalence theorem will be proved once it is shown that (Extension Theorem): Every isotopy $f_t$ is covered by an ambient isotopy $F_t$.

The proof of this theorem is the main result of the succeeding chapters.

Lastly, the third equivalence relation: *ambient homeomorphism equivalence.*

DEFINITION 4. $K \underset{h}{\sim} K'$, or K and K′ are ambient-homeomorphism equivalent if there is an orientation preserving combinatorial homeomorphism

$$h : E^r \to E^r$$

such that $h : K \approx K'$.

On the face of it, this last equivalence relation seems weaker than the others — thus:

$$K \underset{a}{\sim} K' \Rightarrow K \underset{h}{\sim} K'.$$

## § 4. The stability of combinatorial imbeddings.

Let the usual metric be placed on the set M of all combinatorial maps

$$\varphi : K \to B^r \subseteq E^r$$

$$\delta(\varphi, \psi) = \max_{v \in V(K)} || \varphi(v) - \psi(v) ||$$

where V(K) is the set of vertices of K. Let $M = N \cup S$, where N is the subset of imbeddings, and $S = M - N$.

LEMMA 1. There is a continuous function $\rho$ on M with the properties:

(i) $\rho(m) \geq 0$, $\rho(m) > 0 \Leftrightarrow m \in N$.

(ii) If $\varphi$ is an imbedding (i.e. $\varphi \in N$) and $\varphi' \in M$ such that $||\varphi'(v) - \varphi(v)|| < \rho(\varphi)$, for all $v \in V(K)$ then $\varphi'$ is again an imbedding.

(iii) $\rho$ is the maximal function possessing properties (i), (ii).

PROOF. The proof is immediate once one proves:

LEMMA 2. S is compact. Which is trivial, for M is clearly compact and S closed. Then take $\rho(\varphi)=\delta(\varphi, S)$, and it is again immediate that $\rho$ satisfies the requirements of lemma 1.

## § 5. Demonstration of the Equivalence Theorem.

THEOREM. Any isotopy $f_t$ is covered by an ambient isotopy $F_t$.

The nature of the proof is to replace $f_t$ by chains of more restricted kinds of isotopies, which reduces the problem to finding ambient isotopies covering these special isotopies. One proceeds to solve the problem for that special class.

DEFINITION 5. A *perturbation isotopy* $\varphi_t$ is an isotopy such that:

$$||\varphi_1(v)-\varphi_0(v)||<\rho(\varphi_0) \text{ for all } v\in V(K).$$

DEFINITION 6. A *simple isotopy* $\varphi_t$ is an isotopy such that $\varphi_t$ is constant on every vertex in V(K), save one, $v$ and the image of $v$ under $\varphi_t$ is a line segment in $E^r$.

LEMMA 3. Any isotopy $f_t$ between K and K′ can be replaced by a chain of perturbation isotopies $f_t^{(1)}, \ldots, f_t^{(\nu)}$. That is: $f_t^{(i)}$ is an isotopy between $K^{(i-1)}$ and $K^{(i)}$ where:

$$K^{(0)}=K$$
$$K^{(\nu)}=K'.$$

LEMMA 4. Any perturbation isotopy $\varphi_t$ can be replaced by a chain of simple isotopies

$$\varphi_t^{(1)}, \ldots, \varphi_t^{(\mu)}$$

which gives us:

LEMMA 5. Any isotopy $f_t$ may be replaced by a chain of simple isotopies:

$$f_t^{(1)}, \ldots, f_t^{(\nu)}$$

Thus our original theorem reduces to the relatively

LOCAL PROBLEM: Given a simple isotopy $f_t$, find an ambient isotopy $F_t$ covering it.

Clearly the solution of the local problem coupled with Lemma 5 provides a proof of the equivalence theorem.

## § 6. Reduction to perturbation isotopies (Proof of Lemma 3).

Define $\beta(t)=\rho(f_t)$, where $\rho$ is as in Lemma 1. Then $\beta$ is continuous and positive on I and hence it has a minimum M.

LEMMA 6. One may partition the interval I into

$$0=x_0, x_1, \ldots, x_\nu=1$$

so finely that $||f_{x_i}(v)-f_{x_{i+1}}(v)||<M$ for all $v\in V(K)$ and all $i=0, \ldots, \nu-1$. And therefore we would have

(A) $$||f_{x_i}(v)-f_{x_{i+1}}(v)||<\rho(f_{x_i}).$$

The proof of this comes simply from the continuity of $f_t$ in $t$. Define

$$f_t^{(i)}=f_{x_i+t(x_{i+1}-x_i)}$$

and the chain of isotopies $f_t^{(1)}, \ldots, f_t^{(\nu)}$ can replace $f_t$ as an isotopy between K and K'.

Also, (A) becomes: $||f_0^{(i)}(v)-f_1^{(i)}(v)||<\rho(f_0^{(i)})$ for all $v\in V(K)$.

Conclusion. Each $f_t^{(i)}$ is a perturbation isotopy.

## § 7. Reduction to simple isotopies (Proof of Lemma 4).

Let, then, $f_t$ be a perturbation isotopy between K and K'. Order the vertices $\{v_1, \ldots, v_j, \ldots, v_n\}\in V(K)$. I'm going to define a chain of complexes

$$K=K_0,\ K_1, \ldots, K_n=K'$$

and a chain of simple isotopies

$$f_t^{(i)} \qquad i=1, \ldots, n$$

such that $f_t^{(i)}$ is an isotopy between $K_{i-1}$ and $K_i$. Define $K^{(i)}$ to be the image of K under the piece-wise linear map $\mu^{(i)}$ which acts in this way on vertices:

$$\mu^{(i)}(v_j)=\begin{cases} v_j & \text{if } j>i \\ v_j' & \text{if } j\leqslant i\end{cases}$$

Let $f_t^{(i)}$ be the simple isotopy which acts as follows on the vertices:

$$\{v_1', \ldots, v_{i-1}', v_i, \ldots, v_\nu\}=V(K^{(i)})$$
$$f_t^{(i)}(v_j')=v_j' \qquad j<i$$
$$f_t^{(i)}(v_j)=v_j \qquad j>i$$
$$f_t^{(i)}(v_i)=(1-t)v_i+tv_i'.$$

In order to show that:

Lemma 7. $f_t^{(i)}$ is an isotopy between $K^{(i-1)}$ and $K^{(i)}$,
it remains to show :

Lemma 8. $f_t^{(i)}$ is a combinatorial homeomorphism for each $t$.

Proof. Let $f_0 : K\to E^r$ be the simplicial imbedding which is the injection of K in $E^r$. Make $\sigma_t^{(i)}: K\to E^r$ as follows :

$$\sigma_t^{(i)}: v_j\to\begin{cases} v_j' & j<i \\ (1-t)v_j+tv_j' & j=i \\ v_j & j>i\end{cases}$$

Then $f_t^{(i)}\mu^{(i)}=\sigma_t^{(i)}$.

And so $f_t^{(i)}$ will be a combinatorial homeomorphism if $\sigma_t^{(i)}$ is.

But
$$\delta(\sigma_t^{(i)}, f_0) \leqslant \max_j ||v_j' - v_j||$$
$$\delta(\sigma_t^{(i)}, f_0) \leqslant \max_{v \in V(K)} ||f_1(v) - f_0(v)||$$
$$\leqslant \rho(f_0).$$

The last inequality occurs since $f_t$ is simple. Thus, by definition of $\rho$, $\sigma_t^{(i)}$ is a combinatorial homeomorphism, which proves the lemma.

## § 8. Solution of the Local Problem.

PROBLEM. If $f_t$ is a simple isotopy between K and K′, find an ambient isotopy $F_t$ covering $f_t$.

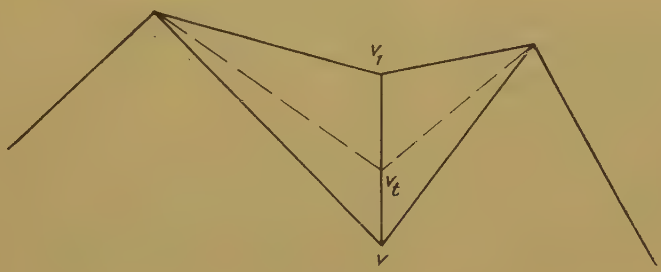


Fig. 1

Terminology: Let $v$ be the vertex of K that $f_t$ moves, $f_t(v) = v_t$. Call V the directed line generated by the vector $\vec{vv_1}$. Call $V_x$ the unique line through $x \in E^m$ parallel to V and $h: E^m \to V$ the map obtained by projecting $E^m$ to V. Call H the hyperplane consisting of the « zeros » of $h$.

$$\begin{array}{l} E^m \xrightarrow{h} V \\ \downarrow \pi \\ H \end{array}$$

Now, since [fig. 1] $f_t$ is the identity except on St $v$, we would like to enclose St $v$ in some region $\Omega$ so that we may define $F_t$ to be the identity in the complement of $\Omega$, and so that $\Omega$ would lend itself nicely to the construction of $F_t$ on it.

## § 9. The region $\Omega$.

There are four properties I shall require:

(i) $\Omega$ is a closed neighborhood of int St $v$, and a finite subcomplex of $E^m$.
(ii) $\Omega \cap K = \mathrm{St}\, v$.
(iii) If $x \in \Omega$, $V_x \cap \Omega$ consists of a single interval.
(iv) If $x \in \mathrm{St}\, v - \partial \mathrm{St}\, v$, $x \subset \mathrm{int}\, V_x \cap \Omega$.
If $x \in \partial \mathrm{St}\, v$, $V_x \cap \Omega = \{x\}$.

Call $\pi(\Omega)=\Omega^*$ and $\widetilde{\Omega}=\Omega^*\times I$. Let $I_x=V_x\cap\Omega$ for $x\in\Omega$, and $I_{\omega^*}=I_\omega$ for $\pi(\omega)=\omega^*\in\Omega^*$.

If $I_0$ is a line segment in $E^r$, let $M(I_0)$ be the simplicial complex of all simplicial homeomorphisms of $I_0$ leaving endpoints fixed. There is a chosen element in $M(I_0)$ (denoted $\iota$), namely the identity homeomorphism.

There is a natural map

$$\eta : M(I)\to M(I_0)$$

which is a homeomorphism if $I_0$ is of positive length. If $I_\omega$ is as above, with $\omega\in\Omega^*$, let's denote this natural map by $\eta_\omega$

$$\eta_\omega : M(I)\to M(I_\omega).$$

Define $\tau:\widetilde{\Omega}\to\Omega$ as follows:

$$\tau(\omega, t)=\eta_\omega(t).$$

Then I require that $\tau$ be a simplicial map :

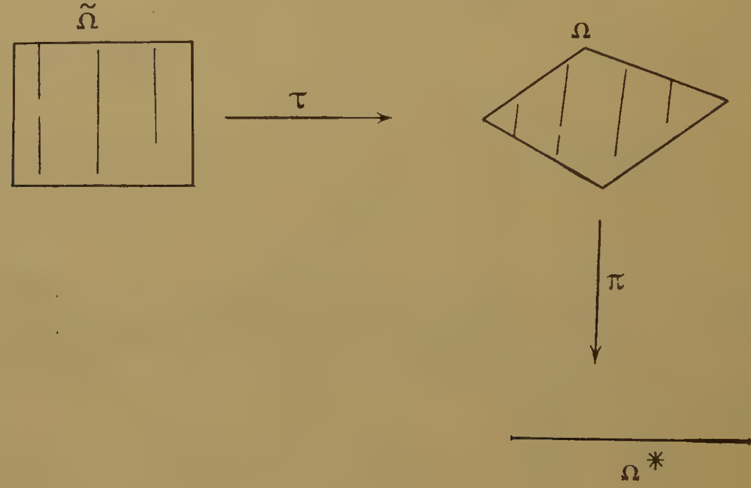


Fig. 2

As a consequence of the axioms, we may deduce these properties of $\tau$ :

*a)* $\tau|\widetilde{\Omega}-(\partial\Omega^*\times I)$ is a homeomorphism.

*b)* $\tau(\omega^*, t)$ is the unique point $\omega\in\Omega$ lying over $\omega^*$, if $\omega^*\in\partial\Omega^*$.

*c)* Let $a(x)$, $b(x)$ be functions $a, b:\Omega\to V$ which assign to each $x\in\Omega$ the upper and lower endpoints (respectively) of the interval $V_x\cap\Omega$. Then $a$ and $b$ are combinatorial functions on $\Omega$.

## § 10. Construction of $\Omega$.

Let O be a finite subcomplex of $E^m$ and a small closed neighborhood of $\overrightarrow{vv_1}$ such that the corresponding axioms (iii) and (iv) hold for it. Moreover, find it small enough so that

$$\Omega_0=J(O,\partial\,\mathrm{St}\,v)$$

has the property that

$$\Omega_0 \cap K = \mathrm{St}\, v.$$

(That such an O can be found is an immediate consequence of the stability lemma.)

LEMMA 9. Any such $\Omega_0$ satisfies the four properties above.
(The proof is mere verification.) So, fix some such $\Omega$.

## § 11. The problem remaining.

We must define $F: I \times E^m \to E^m$ covering $f: I \times K \to E^m$. Since $f_t$ is the identity in the complement of $\Omega$, we may choose $F_t$ to be the identity on $E^m - \Omega$ (i.e. define) :

$$F: I \times (E^m - \Omega) \to E^m - \Omega$$

as $$F: (t, x) \to x.$$

And it remains to define the isotopy

$$F: I \times \Omega \to \Omega$$

such that

*a)* $F_t | \partial\Omega = 1$.
*b)* $F_t$ covers $f_t$ ; i.e. $F_t(x) = f_t(x) = x_t$, if $x \in \mathrm{St}\, v$.

## § 12. V-homeomorphisms and V-isotopies.

DEFINITION 7. A *full subcomplex* $N \subseteq \Omega$, is one such that if $n \in N$, $I_n \subseteq N$ (i.e. $N = \pi^{-1}\pi N$).

DEFINITION 8. A (combinatorial) V-*homeomorphism* $\varphi$ of a full subcomplex $N \subseteq \Omega$ onto itself is one such that:

*a)* $\varphi$ leaves the endpoints of $I_n$ fixed (for all $n \in N$).
*b)* $\varphi$ satisfies the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & N \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \Omega^* & \end{array}$$

(Equivalently, $\varphi(I_x) \subseteq I_x$ for all $x \in N$.) A V-*isotopy* is an isotopy which is a V-homeomorphism at each stage.

LEMMA 10. Any V-isotopy $\varphi_t : \Omega \to \Omega$ must leave $\partial\Omega$ fixed.

PROOF. If $\varphi$ is a V-homeomorphism $\varphi: \Omega \to \Omega$, then $\varphi$ leaves $\partial I_x$ fixed, and $\partial\Omega = \bigcup_{x \in \Omega} \partial I_x$. Define $H_v(N)$ to be the set of all combinatorial V-homeomorphisms of N, given the topology it inherits as a subset of M(N, N). There is a chosen element in $H_v(N)$ which I shall denote by 1. It is the identity V-homeomorphism. Let $I_v(N)$ be the topological space consisting of all V-isotopies of N, i.e. all paths in $H_v(N)$ beginning at 1. (By a *path* in $H_v(N)$, I shall always mean one which begins at 1.)

If N and N′ are full subcomplexes $N \subseteq N'$, there are natural restriction maps

$$\rho : I_v(N') \to I_v(N)$$
$$\rho : H_v(N') \to H_v(N)$$

defined in the obvious manner.

## § 13. A V-isotopy F′ covering *f*.

Let N be the full subcomplex of Ω 'generated' by St $v$

$$N = \bigcup_{x \in St\, v} I_x.$$

Define the V-isotopy $F'_t : N \to N$ as follows: for every $I_x$, let $a(x)$, $b(x)$ be its endpoints, as before.

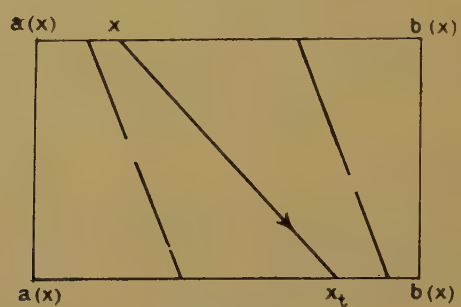


Fig. 3

Define $F'_t$ on $I_x$ by

$$F'_t : a(x) \to a(x)$$
$$F'_t : b(x) \to b(x)$$
$$F'_t : x \to f_t(x) = x_t$$

and extend to $I_x$ by linearity (on each of the subintervals $[a(x), x]$ and $[x, b(x)]$.) $F'_t$ is combinatorial since $a$ and $b$ are combinatorial functions and therefore a V-isotopy. Lastly, $F'_t | St\, v = f_t$, so $F'_t$ covers $f_t$.

Lemma 10 implies that any V-isotopy $F_t$ which extends $F'_t$ must satisfy *a*), *b*) of paragraph 12, and hence would yield the ambient isotopy sought for. After these remarks, it is clear that a solution to the local problem is an immediate corollary to:

*(Extension Lemma):* Let N be a full subcomplex of Ω. Then any V-isotopy of N extends to a V-isotopy of Ω ; or: $\rho : I_v(\Omega) \to I_v(N)$ is onto.

I devote myself, therefore, to a proof of the above.

## § 14. The spaces $X_v(L)$.

Let $EM(I_0)$ be the set of all polygonal paths in $M(I_0)$ beginning at 1. $EM(I_0)$ is endowed with the structure of a simplicial complex in a natural way. (This is standard. It is an infinite simplicial complex given the weak topology.)

LEMMA 11. $EM(I_0)$ is contractible.

And we have the following lemma:

LEMMA 12. $M(I_\omega)$ is a single point $\iota$, if and only if $\omega \in \partial\Omega^*$.

Let $L \subseteq \Omega^*$ be a subcomplex.

$$\begin{tikzcd} & M(I)\times\Omega^* \arrow[dr, "P"] & \\ L - L\cap\partial\Omega^* \arrow[ur, "K"] \arrow[rr] & & \Omega^* \end{tikzcd}$$

Let $X_v(L)$ be the set of all combinatorial cross-sections K over the finite open complex

$$L - L\cap\partial\Omega^*.$$

$X_v(L)$ is given the topology it inherits as a subset of $M(L - L\cap\partial\Omega^*, M(I)\times\Omega^*)$. There is a chosen cross-section in any $X_v(L)$, which consists of the identity function $\iota: I_l \to I_l$ for any $l \in L$. This I call the identity cross-section $(\iota_L)$. Since $X_v(L)$ is a topological space, I may speak of paths on it. And again: by a path in $X_v(L)$ I'll always mean one beginning at $\iota_L$.

## § 15. The Cross-Section Extension Lemma.

I state it in a particularly useful way:

Any path of cross-sections $K_t$ in $X_v(L)$ for $L \subseteq \Omega^*$ may be extended to a path of cross-sections $\overline{K}_t$ in $X_v(\Omega^*)$. (Equivalently: there is a path $\overline{K}_t$ in $X_v(\Omega^*)$ such that $\overline{K}_t | L = K_t$, for each $t$.)

PROOF. Let $L \subseteq \Omega^*$, and $EM(I)$ as before. Thus, paths of cross-sections $K_t$ in $X_v(L)$ correspond simply to cross-sections in the bundle:

$$\begin{tikzcd} & EM(I)\times\Omega^* \arrow[dr, "P"] & \\ L - L\cap\partial\Omega^* \arrow[ur, "K"] \arrow[rr] & & \Omega^* \end{tikzcd}$$

Therefore the question of extension of paths in $X_v(L)$ to $X_v(\Omega^*)$ is a question of extension of K to $\overline{K}$:

$$\begin{tikzcd} & EM(I)\times\Omega^* \arrow[dr, "P"] & & \\ L - L\cap\partial\Omega^* \arrow[ur, "K"] \arrow[r] & \Omega^* - \partial\Omega^* \arrow[u, "\overline{K}"] \arrow[r] & \Omega^* \end{tikzcd}$$

But contractibility of $EM(I)$ gives all. (A standard reference to such lemmas on the extension of cross-sections in fibre bundles is: Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press.)

## § 16. The Liaison.

PROPOSITION. If $N \subseteq \Omega$ is a full subcomplex, and $N^* \subseteq \Omega^*$ its image under $\pi$, then there is a homeomorphism $\lambda: X_v(L^*) \approx H_v(L)$ such that $\lambda(\iota_{L^*}) = \iota_L$, and

$$\eta_\omega[\xi(\omega)] = \lambda(\xi) | I_\omega$$

for $\xi$ a cross-section in $X_v(L^*)$.

PROOF.

1) To any cross-section $K \in X_v(L^*)$ I may associate a combinatorial V-homeomorphism $\bar{K} : (L^* - L^* \cap \partial\Omega^*) \times I \rightarrow (L^* - L^* \cap \partial\Omega^*) \times I$ where $P = L^* - L^* \cap \partial\Omega^*$ is considered as a subset of $\widetilde{\Omega}$. Define $\bar{K}(l \times I) = K(l)(I)$, where $K(l) \in M(I)$ is considered as a function of I. If K is a combinatorial cross-section with respect to the simplicial structure on $M(I) \times \Omega^*$, then $\bar{K}$ is a combinatorial V-homeomorphism.

2) To any combinatorial V-homeomorphism $\varphi : P \rightarrow P$, $P \subseteq \widetilde{\Omega} - I \times \partial\Omega^*$ one may associate a combinatorial V-homeomorphism $\varphi'$ making the diagram below commutative:

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow \tau & & \downarrow \tau \\ \tau(P) & \xrightarrow[\varphi']{} & \tau(P) \end{array}$$

(since $\tau$ is a homeomorphism on $\widetilde{\Omega} - I \times \partial\Omega^*$). Moreover, $\varphi'$ is extendable to a V-homeomorphism $\varphi''$

$$\varphi'' : \tau(P) \cup \pi^{-1}(\partial\Omega^*) \rightarrow \tau(P) \cup \pi^{-1}(\partial\Omega^*)$$

(in fact, since $I_\omega$ is a single point for $\omega \in \partial\Omega^*$, a V-homeomorphism *must* be the identity map on $\pi^{-1}(\partial\Omega^*)$) and so we must define

$$\varphi'' | \pi^{-1}(\partial\Omega^*) = 1.$$

It is evident that $\varphi''$ so defined is a V-homeomorphism on the larger set.

3) Combining 1) and 2), one obtains a map

$$\lambda : X_v(L^*) \rightarrow H_v(L)$$
$$\lambda(K) = (\bar{K})'' | L$$

(Notice : $\bar{K}$ is defined on $(L^* - L^* \cap \partial\Omega^*) \times I$ and $\bar{K}'$ on $L - L \cap \pi^{-1}\partial\Omega^*$, and finally $\bar{K}''$ on $(L - L \cap \pi^{-1}\partial\Omega^*) \cup \pi^{-1}\partial\Omega^* = L \cup \pi^{-1}\partial\Omega^*$. So we must restrict $\bar{K}''$ back to L.)

It is straightforward that $\lambda$ is a homeomorphism. (The construction of $\lambda^{-1}$ is immediate.)

## § 17. **Conclusion.**

The proof of the extension lemma now follows easily:

LEMMA 12. Any path $F_t$ in $H_v(N)$ extends to a path $\widetilde{f}_t$ in $H_v(\Omega)$ for $N \subseteq \Omega$, a full subcomplex.

PROOF. By the previous paragraph, $\lambda^{-1}(F_t)$ is a path in $X_v(N)$, and by the cross-section extension lemma, it extends to a path $\widehat{\lambda^{-1}F_t}$ in $X_v(\Omega^*)$. Finally, it is evident that

$$\widetilde{f}_t = \lambda(\widehat{\lambda^{-1}F_t}) \in I_v(\Omega)$$

is an extension of $F_t$.

And so, the main theorem follows.

## § 18. A Strengthening of The Main Theorem.

To avoid any undue complication in the proof of the main theorem, I stated it in its simplest, and therefore least useful, form. For later applications, I will need a strengthened statement of the theorem, which 'globalizes' the range space.

By a homogeneous bounded $n$-manifold I shall mean a finite complex, which is, topologically, a bounded $n$-manifold, and which has a group of combinatorial automorphisms which is transitive on interior points, and on each connected component of the boundary.

The strengthened theorem will involve replacing $E^r$ by a general homogeneous manifold, W. I must stipulate, therefore, what I mean by an isotopy $f_t : K \to W$.

Definition. Let K and W be finite complexes with particular triangulations. Then $\Psi : K \to W$ is a piecewise linear map if simplices of K are mapped linearly into simplices of W.

Definition. An *Isotopy* $f_t : K \to W$ is a continuous family of homeomorphisms of K into W, piecewise linear for a fixed triangulation of K and W (independent, of course, of $t$), where $0 \leq t \leq 1$.

Definition. An ambient isotopy $F_t : W \to W$ is a continuous family of combinatorial automorphisms of W such that $F_0 = 1$, and $F_t | K = f_t$ is an isotopy of K in W. $F_t$ is then said to be an ambient isotopy covering $f_t$.

The Strengthened Version. Let M and W be bounded homogeneous manifolds (possibly not of the same dimensions), and let $M'$, $W'$ be the boundaries of M and W, respectively. Let $f_t$ be an isotopy of M through W. Thus, M is to be regarded as a submanifold of W, and $f_0$ is the inclusion map.

Let us assume that $f'_t = f_t | M'$ is an isotopy of $M'$ through $W'$. Finally, let $F'_t$ be an ambient isotopy of $W'$ covering $f'_t$.

Then, there is an ambient isotopy $F_t$ covering $f_t$ such that $F_t | M' = F'_t$.

I omit a proof of this elaboration. Such a proof may be obtained by merely restating the arguments of the main theorem in this more general setting. It would involve complications only in terminology.

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# ON THE STRUCTURE OF CERTAIN SEMI-GROUPS OF SPHERICAL KNOT CLASSES

*By* Barry MAZUR

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## § 1. Introduction.

The problem of classification of $k$-sphere knots in $r$-spheres is the problem of classifying "knot pairs": $S=(S_1, S_2)$, where $S_2$ is an oriented combinatorial $r$-sphere, $S_1$ a subcomplex of $S_2$ (isomorphic to a standard $k$-sphere), and the pair S is considered equivalent to $S'$ $(S \sim S')$ if there is a combinatorial orientation-preserving homeomorphism of $S_1$ onto $S'_1$ bringing $S_2$ onto $S'_2$.

Thus it is the problem of classifying certain relative combinatorial structures. The set of all such, for fixed $k$ and $r$, will be called $\Sigma^r_k$, and can be given, in a natural manner, the structure of a semi-group. There is a certain sub-semi-group of $\Sigma^r_k$ to be singled out — the semi-group $S^r_k$ of all pairs $S=(S_1, S_2)$ where $S_1$ is smoothly imbedded in $S_2$ (locally unknotted).

In this paper I shall define a notion of equivalence (which I call $*$-equivalence) between knot pairs which is (seemingly) weaker than the equivalence defined above.

Two knot pairs S and $S'$ are $*$-equivalent if (again) there is an orientation-preserving homeomorphism

$$\varphi : S_2 \to S'_2$$

bringing $S_1$ onto $S'_1$. However $\varphi$ is required to be combinatorial (not on all of $S_2$, as before, but) merely on $S^*_2 = S_2 - (p_1, \ldots, p_n)$, where $p_1, \ldots, p_n \in S_2$, where $S^*_2$ is considered as an open infinite complex. Thus $*$-equivalence neglects some of the combinatorial structure of the pair $(S_1, S_2)$. The set of all $*$-equivalence classes of knot pairs forms a semi-group again, called ${}^*\Sigma^r_k$.

Finally the subsemi-group of smoothly imbedded knots in ${}^*\Sigma^r_k$ I call ${}^*S^r_k$. The purpose of this paper is to prove a generalized knot theoretic restatement of lemma 3 in [1].

Inverse Theorem: A knot $S^r_k$ is invertible if and only if it is $*$-trivial.

And in application, derive the following fact concerning the structure of the knot semi-groups:

There are no inverses in ${}^*S^r_k$.

## § 2. Terminology.

My general use of combinatorial topology terms is as in [2]. It is clear what is meant by the "usual" or "standard" imbedding of a $k$-sphere or a $k$-cell in $E^r$. Similarly an unknotted sphere or disc in $E^r$ means one that may be thrown onto the usual by a combinatorial automorphism of $E^r$.

Definition 1. Let $M^k$ be a subcomplex (a $k$-manifold) of $E^r$. Then $M^k$ is *locally unknotted at a point m* $(m \in M)$ if the following condition is met with:

1) There is an $r$-simplex $\Delta^r$ drawn about $m$ so that $\Delta^r \cap M \subset St(m)$, and $\Delta^r \cap M$ is then a $k$-cell $B^k \subset \Delta^r$, and $\partial B^k \subset \partial \Delta^r$.

2) There is a combinatorial automorphism of $\Delta^r$, sending $B^k$ onto the "standard $k$-cell in $\Delta^r$". M is plain *locally unknotted* if it is locally unknotted at all points.

*Semi-Groups:*

All semi-groups to be discussed will be countable, commutative, and possess zero elements.

Definition 2. A semi-group F is *positive* if:

$$X + Y = 0 \text{ implies } X = 0$$

(i.e.if F has no inverses).

Definition 3. A *minimal base* of a semi-group F is a collection $J = (\chi_1, \ldots)$ of elements of F such that every element of F is a sum of elements in J, and there is no smaller $J' \subset J$ with the same property.

Definition 4. A *prime element p* in the semi-group F is an element for which $p = x + y$ implies either $x = 0$ or $y = 0$.

Clearly, if a positive semi-group F possesses a minimal base, that minimal base has to be precisely the set of primes in F, and F has the property that every element is expressible as a finite sum of primes.

Definition 5. An element $x \in F$ is *invertible* if there is a $y \in F$ such that

$$x + y = 0.$$

## § 3. (*)-homeomorphism.

Definition 6. A $(p_1, \ldots, p_n)$-homeomorphism, $h : E^r \to E^r$ will be an orientation preserving homeomorphism which is combinatorial except at the points $p_i \in E^r$. It is a homeomorphism such that $h | E^r - (p_i)$ is a combinatorial map — simplicial with respect to a possibly infinite subdivision of the open complexes involved. When there is no reason to call special attention to the points $p_1, \ldots, p_n$, I shall call such: a (*)-homeomorphism.

DEFINITION 7. Two subcomplexes $K, K' \subset E^r$ will be called *$*$-equivalent* ($K \underset{*}{\sim} K'$) if there is a $*$-homeomorphism $h$ of $E^r$ onto itself bringing K onto K'. (If $h$ is a $(p_i)$-homeomorphism I shall also say $K \underset{(p_i)}{\sim} K'$.) To keep from using too many subscripts, whenever a $(*)$-equivalence comes up in a subsequent proof, I shall act as if it were a $(p)$-equivalence for a single point $p$. This logical gap, used merely as a notation-saving device, can be trivally filled by the reader.

I'll say a sphere knot is *$*$-trivial* if it is $*$-equivalent to the standard sphere.

## § 4. Knot Addition.

There is a standard additive structure that can be put on $\Sigma_k^r$, the set of combinatorial $k$-sphere knots in $E^r$ (two $k$-sphere knots are equivalent if there is an orientation-preserving combinatorial automorphism of $E^r$ bringing the one knot onto the other). (For details see [**2**]).

I shall outline the procedure of "adding two knots" $S_0$, $S_1$. Separate $S_0$ and $S_1$ by a hyperplane H (possibly after translating one of them). Take a $k$-simplex $\Delta_i$ from each $S_i$, $i = 0, 1$. And lead a "tube" from $\Delta_0$ to $\Delta_1$ (by "thickening" a polygonal arc joining a point $p_0 \in \Delta_0$ to $p_1 \in \Delta_1$, which doesn't intersect the $S_i$ except at $\Delta_i$). Then remove the $\Delta_i$ and replace them by the tube $T = S^{k-1} \times I$, where one end, $S^{k-1} \times 0$ is attached to $\partial \Delta_0$ by a combinatorial homeomorphism, and the other $S^{k-1} \times 1$ is attached to $\partial \Delta_1$ similarly. The resulting knot is called the sum: $S_0 + S_1$, and its knot-equivalence class is unique.

If one added the point at infinity to $E^r$, to obtain $S^r$, the hyperplane H would become an unknotted $S^{r-1} \subset S^r$, separating the knot $S_0 + S_1$ into its components $S_0$ and $S_1$. In analytic fashion, then, we can say that a $k$-sphere knot $S \subset S^r$ is *split* by an $S^{r-1} \subset S^r$ if:

1) $S^{r-1} \cap S$ is an unknotted $(k-1)$-sphere knot in S.
2) $S^{r-1}$ is unknotted in $S^r$.
3) $S^{r-1} \cap S$ is unknotted in $S^{r-1}$.

Let $A_0$ and $A_1$ be the two complementary components of $S^{r-1} \cap S$ in S, and let B be an unknotted $k$-disc that $S^{r-1} \cap S$ bounds in $S^{r-1}$. Then $S_0 = A_0 \cup B$, $S_1 = A_1 \cup B$ are knotted spheres again, and clearly $S \sim S_0 + S_1$.

Thus I'll say: $S^{r-1}$ splits S into $S_0 + S_1$ ; if $E_0$ and $E_1$ are the complementary regions of $S^{r-1}$ in $S^r$, I'll refer to $S_1$ as *that « part of S » lying in $E_1$*, and similarly for $S_0$. Working in the semi-group ${}^*\Sigma_k^r$, one can be slightly cruder, and say: *$S^{r-1}$ $*$-splits S* if only 1) and 3) hold. Clearly by [**1**], every $S^{r-1}$ is $*$-trivial in $S^r$.

LEMMA 1: If $S^{r-1}$ $*$-splits S, and $S_0$, $S_1$ are constructed in a manner analogous to the above, then $S \underset{*}{\sim} S_0 + S_1$.

## § 5. The Semi-Groups of Spherical Knots.

This operation of addition, discussed in the previous section, turns $\Sigma_k^r$ into a commutative semi-group with zero. Our object is to study the algebraic structure of the

subsemi-group $S_k^r \subset \Sigma_k^r$ of locally unknotted $k$-sphere knots. Let ${}^*\Sigma_k^r$ be the semi-group of classes of spherical knots under $*$-equivalence. Let $G_k^r \subset \Sigma_k^r$ be the maximal subgroup of $\Sigma_k^r$, that is: the subgroup of invertible knots.

Inverse Theorem: There is an exact sequence

$$0 \to G_k^r \to S_k^r \to {}^*S_k^r \to 0$$

(where ${}^*S_k^r$ is the image of $S_k^r$ in ${}^*\Sigma_k^r$)

or, equivalently, a knot in $S_k^r$ is $*$-trivial if and only if it is invertible.

## § 6. Proof of the Inverse Theorem.

*a)* If S is invertible, then $S \underset{(*)}{\sim} 0$. The proof is quite as in [1]. Let $S + S' \sim 0$. Then consider the knots:

$$S_\infty = S + S' + S + S' + \dots \cup p_\infty$$
$$S'_\infty = S' + S + S' + S + \dots \cup p_\infty$$

(See figure 1)

and notice: (as was done in detail in [1])

$$S_\infty \underset{(p_\infty)}{\sim} 0$$
$$S'_\infty \underset{(p_\infty)}{\sim} 0$$
$$S_\infty = S + S'_\infty$$

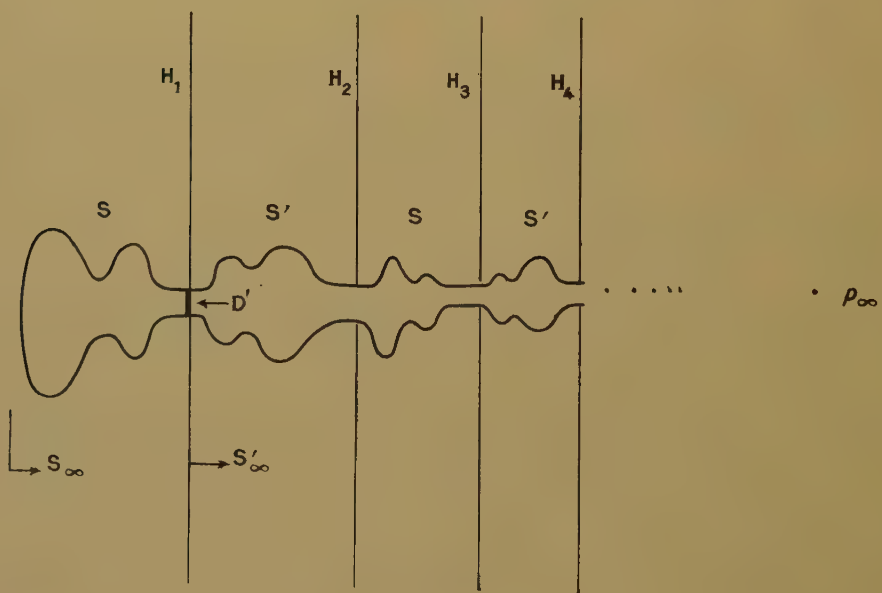


Fig. 1

Lemma 2: There is a $(*)$-homeomorphism

$$f: E^r \to E^r \text{ such that}$$
$$f: S \to S + S'_\infty.$$

PROOF: Let D be the $k$-cell on which the addition of S to $S'_\infty$ takes place. Since $S'_\infty \underset{(p_\infty)}{\sim} 0$, we may transform figure 1 to figure 2 by a $(p_\infty)$-homeomorphism $g$ which leaves everything to the left of the hyperplane $H_1$ fixed, and sends S′ to the "standard $k$-sphere" to the right of $H_1$. (See figure 2.)

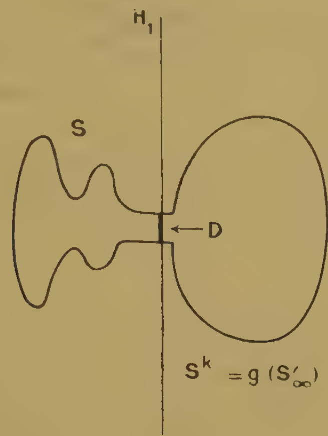


Fig. 2

Then, in figure 2, clearly one can construct an automorphism $f'$ which leaves S fixed and sends D onto $g(S'_\infty)$—int D.

Take $f=g^{-1}f'g$, and $f$ has the properties required, and is a $(*)$-homeomorphism. Therefore, by the above lemma,

$$S \underset{(*)}{\sim} S + S'_\infty = S_\infty \underset{(*)}{\sim} 0$$

and finally:

$$S \underset{(*)}{\sim} 0$$

which proves $(a)$.

*b)* If $S \in S^r_k$ and $S \underset{(p)}{\sim} 0$, then S is invertible.

PROOF: First observe that if $k=r-1$, invertibility of knots is generally true (by [1]), and so we needn't prove anything.

LEMMA 3: If $k<r-1$, and $S \in S^r_k$, $S \underset{(p)}{\sim} 0$ for $p \notin S$, then $S \sim 0$.

PROOF: There is an $r$-cell $\Delta$ containing S but not $p$. Then $f|\Delta$ is combinatorial, and by a standard lemma:

LEMMA 4: If $g:\Delta \to \Delta'$ is a combinatorial homeomorphism of an $r$-cell $\Delta \subset E^r$ to an $r$-cell $\Delta' \subset E^r$, then $g$ can be extended to a combinatorial automorphism of $E^r$ (see [2]). Thus, restrict $f$ to $\Delta$, and extend $f|\Delta$ to a combinatorial automorphism $g$ of $E^r$. This $g$ yields the equivalence $S \sim 0$. Therefore, assume $S \underset{(p)}{\sim} 0$, and $p \in S$.

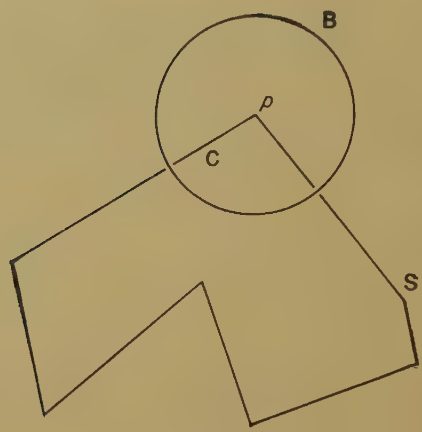


Fig. 3

Let B be a small $r$-cell about $p$, so that $C = B \cap S$ is in $\mathrm{St}(p)$, and hence an unknotted $k$-cell, by the local unknottedness of S. $\partial B \cap S = \partial C$ and $\partial C$ is unknotted in $\partial B$.

Let $f$ be the $(p)$-homeomorphism taking S onto the standard $S^k$.

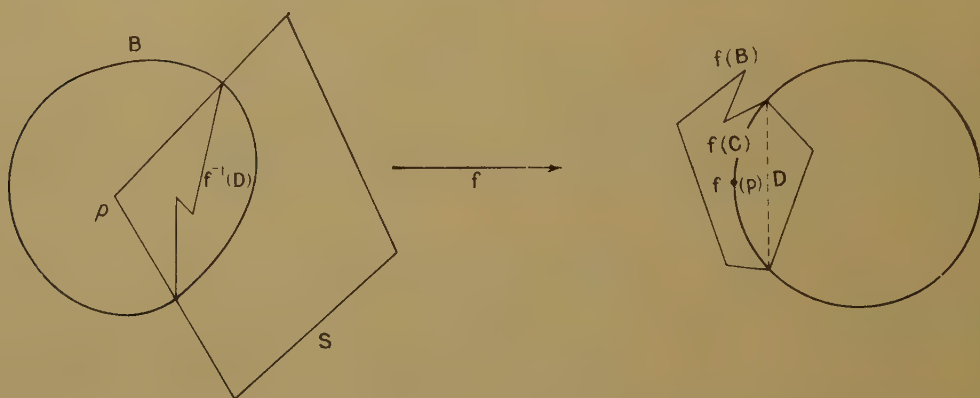


Fig. 4

Now let D be an unknotted disc, the image of a perturbation of $f(C)$ with the properties:

i) $\partial(f(C)) = \partial D$ ;

ii) $\mathrm{int}\ D \subset \mathrm{int}\ B$ ;

iii) $f(p) \notin D$ ;

iv) the knot $K = D \cup (S^k - f(C))$ is still trivial.

Then $f^{-1}$ takes K to a knot $K' = f^{-1}(K)$, split by $\partial B$ into the sum:

$$K' = S + S'$$

where S is the knot lying in the exterior component of $\partial B$, and $S'$ in the interior.

But $K \sim o$, and $K' \underset{f(p)}{\sim} K$ where $f(p) \notin f(K)$, therefore by lemma 3, $f(K) \sim K$. So:

$$S + S' \sim f(K) \sim K \sim o,$$

and $S'$ is invertible.

Corollary: ${}^*S^r_k$ is a positive semi-group.

So we have that ${}^*S^r_k$ is precisely $S^r_k$ « modulo units ».

## § 7. Infinite Sums in ${}^*\Sigma^r_k$.

Let $X_i$, $i = 1, \ldots,$ be knots representing the classes $\chi_i \in \Sigma^r_k$. Define $\sum_{i=1}^{\infty} X_i$ to be the infinite one point compactified sum of the knots $X_i$, in that order (figure 5).

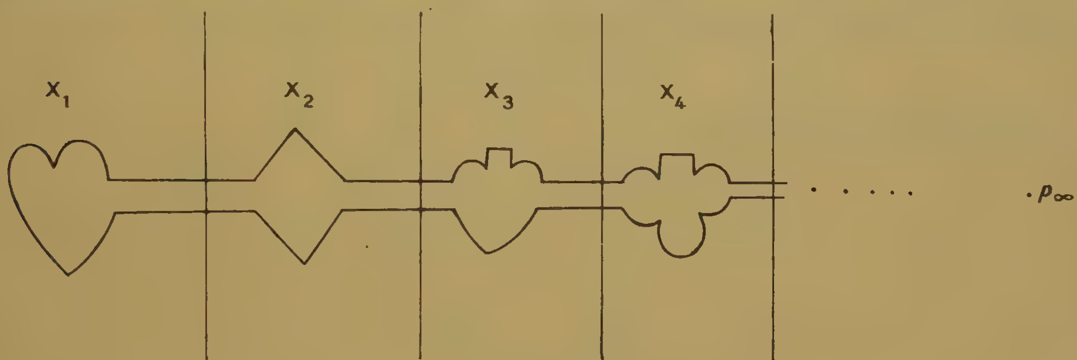


Fig. 5

As it stands, $X = \sum_{i=1}^{\infty} X_i$ will not represent a knot in $\Sigma^r_k$, because X is not combinatorially imbedded (at $p_\infty$).

Definition 8. $\sum_{i=1}^{\infty} X_i = X$ *converges* if there is a $(p_\infty)$-homeomorphism $H : X \to Y$, where Y is combinatorially imbedded. In that case, the knot class $y \in {}^*\Sigma^r_k$ is uniquely determined by the $X_i \in \Sigma^r_k$, and I shall say $\sum_{i=1}^{\infty} \chi_i = y$.

If $\sum_{i=1}^{\infty} \chi_i$ is in ${}^*S^r_k$, I'll say that $\sum_{i=1}^{\infty} \chi_i$ *converges in* ${}^*S^r_k$.

Theorem 1. If $\sum_{i=1}^{\infty} \chi_i$ converges in ${}^*S^r_k$, then it does so finitely. That is, there is an N such that

$$\chi_i \underset{*}{\sim} o, \qquad i > N.$$

Proof: Notice that by the inverse theorem, there are no inverses in ${}^*S^r_k$.

Let $X = \sum_{i=1}^{\infty} X_i$, and $H : X \to Y$ where Y is a subcomplex of $E^r$ and H a $(*)$-homeomorphism.

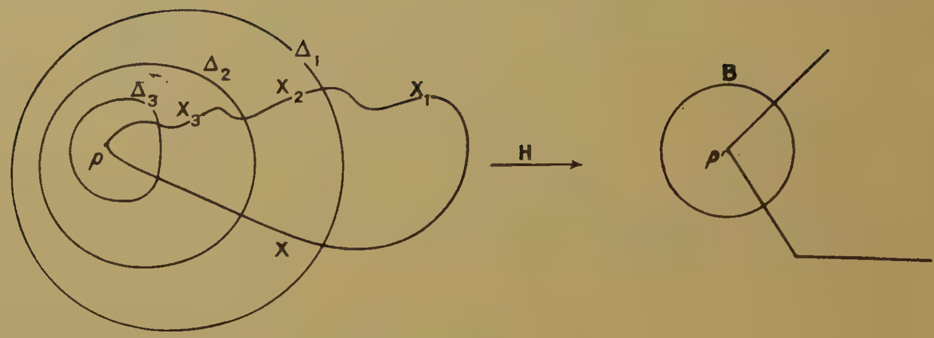


Fig. 6

Let B be a ball about $p'$ such that $B \cap Y$ is a disc in $St(p')$, and by the local unknottedness of Y, $\partial B$ splits Y into two knots,

$$Y = Y^{(1)} + Y^{(2)}$$

where $Y_1 \subset B$ is trivial, and $Y \sim Y_2$.

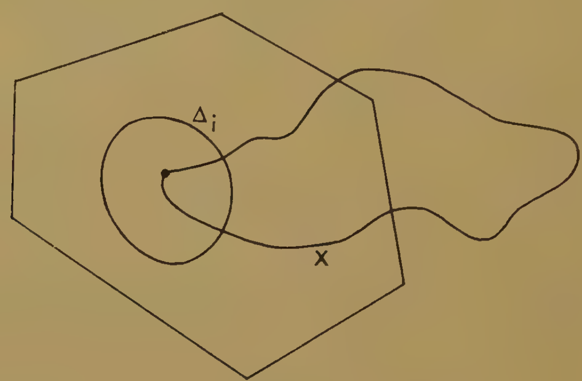


Fig. 7

Now transform the situation by $H^{-1}$. Let $B' = H^{-1}(B)$, and we have that $\partial B'$ $*$-splits X into:

$$X \underset{*}{\sim} X^{(1)} + X^{(2)}$$

and H yields the $*$-equivalences:

$$X^{(1)} \underset{*}{\sim} Y^{(1)} \sim 0$$

$$X^{(2)} \underset{*}{\sim} Y^{(2)}$$

Find an $i$ so large that $\Delta_i \subset \text{int } B'$. Then $\partial\Delta_i$ splits $X^{(1)}$ further:

$$X^{(1)} \sim X^{(3)} + X^{(4)}$$

where $X^{(3)}$ is the part of $X^{(1)}$ lying in $\Delta_i$. But then, by figure 6, $X^{(3)}$ is nothing more than:

$$X^{(3)} \sim \sum_{j=i}^{\infty} X_j.$$

Passing to equivalence classes in ${}^*S_k^r$, one has:

$$\chi^{(3)} + \chi^{(4)} = 0$$

$$\chi^{(3)} = \sum_{j=i}^{\infty} \chi_j$$

(where $x$ the $*$-equivalence class of X). But repeated application of the fact that ${}^*S_k^r$ has no inverses yields $\chi_j = 0$ for $j \geq i$, which proves the theorem.

REFERENCES

[1] B. Mazur, On Imbeddings of Spheres, *Acta Mathematica* (to appear).
[2] V. K. A. M. Guggenheim, Piece wise Linear Isotopy, *Journal of the London Math. Soc.*, vol. 46.

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# ORTHOTOPY AND SPHERICAL KNOTS

*By* Barry MAZUR

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The classical knot theory analyzes imbeddings of the one-sphere in three-space, and its methods conceivably apply, and generalize, to yield some information concerning $n$-sphere knots in $n+2$ space. Most crucial to the theory is the fact that in this range of dimensions, the complementary space of the knot is a delicate indicator of the equivalence class of the knot. (In the classical situation the fundamental group of the complementary space is enough to determine whether the knot is trivial.)

Deviate, however, from this range of dimensions: $n$-sphere knots in $n+2$ space, and the homotopy type of the complementary space gives absolutely no information. It is independent of the knot class.

Concerning ranges of dimension other than "$n$ in $n+2$" very little is known. For instance: It is unknown whether there are any non-trivial imbeddings of spheres in euclidean space, where the codimension of the sphere is different from 2.

There are, however, certain negative results if the dimension of the ambient euclidean space is sufficiently large with respect to the dimension of the sphere.

There is a theorem of Guggenheim:

Theorem: Any two imbeddings of $K^n$ in $E^r$ are isotopic if $n$ is the dimension of K, and

$$r \geq 2n+2.$$

And then, for the case of spheres, there is refinement, due (independently) to Milnor and Wu (unpublished):

Theorem: If $K^n$ is $S^n$, then the above theorem can be improved to read:

$$r \geq 2n+1.$$

The main theorem of this paper is along the lines of these two theorems. It says that for a broad range of dimensions $(r \geq (3n+5)/2)$ any $n$-sphere knot in $E^r$ (fulfilling a certain requirement of local smoothness) is $*$-trivial. (For a definition and treatment of $*$-triviality, see [**2**]. Briefly, a spherical knot is $*$-trivial if there is a homeomorphism of euclidean space onto itself sending the knot onto the standard imbedding of the sphere, such that the homeomorphism is combinatorial except possibly at one point.)

The paper is divided in two parts, the first being devoted to a study of orthotopy,

and general position techniques. The second part uses this theory to prove the main theorem.

I am most thankful to Prof. Milnor who allowed me to see his manuscript.

## § 1. Terminology.

I rely upon [3], for general terminology, and permit myself the following loose usage: Homeomorphism will always mean combinatorial homeomorphism ; a subcomplex $A \subset E^r$ will mean that A is a complex whose imbedding homeomorphism

$$i : A \to E^r$$

is piecewise linear ; the "standard" $k$-sphere, $S^k \subset E^r$ is an image of $S^{r-1} \cap L^{k+1}$ under affine transformation, where $L^{k+1}$ is a $(k+1)$-dimensional subvector space of $E^r$, and $S^{r-1}$ is the unit sphere in $E^r$. The metric I shall place on $E^r$ is :

$$||x|| = \max |x_i| \quad \text{if} \quad x = (x_1, \ldots, x_r), \; x_r \in R.$$

A *homogeneous n-manifold* M will refer to a finite complex which is topologically an $n$-manifold, for which A(M), the group of combinatorial automorphisms of M, is transitive (i.e. usually called a combinatorial $n$-manifold).

By a *regular neighborhood*, A, of N(A), a subcomplex of B, I shall mean the closure of the second regular neighborhood (as defined in page 72 of Eilenberg-Steenrod ; I do not mean what they mean by regular neighborhood).

Let $S \subset E^r$ be any set. Then R(S) is the linear manifold spanned by S:

$$R(S) = (x \in E^r \,|\, x = \alpha s_1 + (1-\alpha) s_2, \; \alpha \in R, \; s_1, s_2 \in S).$$

If $K^n \subset L^m$ is an $n$-dimensional complex in an $m$-dimensional complex, then the codimension of K in L is:

$$\text{cod } K = m - n.$$

If X is a metric space (i.e. if $X = E^r$) then $d(A, B)$ is the distance from A to B, where A and B are compact sets. Also, let $p \in E^r$, then $B_\varepsilon(p) = (x \in E^r \,|\, d(x, p) \leq \varepsilon)$.

Define $E^r_\pm \subset E^r$ to be

$$E^r_+ = [(x_1, \ldots, x_r) \in E^r \,|\, x_r \geq 0]$$
$$E^r_- = [(x_1, \ldots, x_r) \in E^r \,|\, x_r \leq 0]$$

and they are called the upper and lower half-planes, respectively.

## § 2. The definition of knot equivalence.

I will say that two subcomplexes $K \subset E^r$, $K' \subset E^r$ are *equivalent* (and I denote this by: $K \sim K'$) if there is a homeomorphism

$$T : E^r \to E^r$$

such that

$$T : K \to K'$$

is a homeomorphism of K onto K'. Thus the question of classification of equivalence classes of imbeddings of K in $E^r$ is the classification of the combinatorial type of the "relative" manifolds ($E^r$, K). Equivalence is just what was called an ambient homeomorphism equivalence in [3]. A fact used most frequently in this paper is an immediate corollary of the main theorem of [3]:

THEOREM 1. If $f_t : K \to E^r$ is an isotopy between K and K' then $K \sim K'$.

## § 3. Virtual Dimension.

In proving and applying many of the "general position" lemmas that will be developed (all of which involve consideration of the dimension of complexes), I will use a systematic and obvious alteration of the concept of dimension (virtual dimension) which will never be larger than the usual dimension of K (most often smaller), thereby "strengthening" those general position arguments which depend upon the dimension of K being small.

DEFINITION 1. Let L, $K \subseteq E^r$ be two complexes in $E^r$. I will say that the virtual dimension of K with respect to L is less than or equal to $k$ (in symbols: virt $\dim_L(K) \leq k$) if: There is a $k$-dimensional complex P, and a sequence of regular neighborhoods of P: $M_0 \supset M_1 \supset \ldots$, such that $\bigcap_{i=0}^{\infty} M_i = P$, such that there is a homeomorphism of $E^r$ leaving L fixed which brings K into any $M_i$.

If N is a regular neighborhood of K, and L is $E^r$—N, our notation can be reduced to: virt $\dim_L K =$ virt dim K. Notice:

$$\text{virt } \dim_L K \leq \dim K,$$

and that the following three conditions are equivalent:

(i) virt $\dim_L K = 0$
(ii) virt $\dim_K L = 0$
(iii) K and L are *unlinked.*

The generalization of results stated in terms of dimension to corresponding results stated in terms of virtual dimension, being rather straightforward, I henceforth adopt the policy of proving all results merely for dimension, and leaving the transition to virtual dimension to the reader.

For later application of virtual dimension I point out an obvious lemma:

LEMMA 1. Let U be a regular neighborhood of V. Then

$$\text{virt dim } U \leq \dim V$$

## § 4. The Problems of Local Smoothness.

The most obvious distinction between combinatorial imbeddings and differentiable ones is the possibility of a certain local unsmoothness to occur in the combinatorial

situation which has no counterpart in the differentiable. The simplest example of these phenomena is obtained by taking a knotted $S^1 \subset E^3$, and considering $E^3 \subset E^4$ imbedded as a linear hyperplane. Then take a point $P \in E^4$ outside of $E^3$, and draw all line segments from $p$ to points on $S^1 \subset E^3$. The locus, $D^2 \subset E^4$, of these line segments is a combinatorial 2-cell, which is "knotted" in $E^4$. A clear manifestation of its "knottedness" is: If $B = B(p)$ is any small ball drawn about $p$, and $S = \partial B \cap D^2$, then $S$ is homeomorphic with $S^1$, and $S \subset \partial B$ is knotted. Such a phenomenon could not occur if $D^2$ were a differentiable disc imbedded in $E^4$. I should like to rule out the possibility of severe local unsmoothness in the imbeddings which I consider.

Situations such as the above are eliminated by requiring that the imbedding be *locally unknotted* (for the definition ; see [**2**]).

More convenient for the purpose of this paper is a different local smoothness condition:

Definition 2. A subcomplex $K \subset E^r$ is called *homogeneously imbedded* (or just: *homogeneous*) if for any continuous family of homeomorphisms

$$P_t : K \to K$$

such that $P_0$ is the identity, and for any regular neighborhood N of K, there is a homeomorphism

$$P : E^r \to E^r$$

such that $P|E^r - N = 1$ and $P|K = P_1$.

I don't know whether or not the two conditions local unknottedness and homogeneity are the same. That neither restriction is very restrictive may be seen by the following heuristic statement which would lead to unwarranted digression, if I were to attempt to make it precise. Let $\Sigma$ be a combinatorial imbedding of a $k$-sphere in $E^r$ which is a "very close approximation" to S, a differentiable imbedding. Then $\Sigma$ is both homogeneous and locally unknotted.

## § 5. The knot Semi-Groups.

There is a natural additive structure to the set of all equivalence classes of $n$-manifolds combinatorially embedded in $E^r$ (see [**1**] for precise definition), where if $M_0$ and $M_1$ are two knotted $n$-manifolds in $E^r$, $M_0 + M_1$ is essentially obtained by displacing the $M_i$ so that one lies in the upper half-plane and the other in the lower half-plane, then join the $M_i$ by removing an $n$-simplex $\Delta_i$ from each, and attaching a tube, $S^{n-1} \times 1$ such that

$$S^{n-1} \times 0 = \partial\Delta_0 \subseteq M_0$$
$$S^{n-1} \times 1 = \partial\Delta_1 \subseteq M_1.$$

This process is standard, and I call the resulting semi-group of knots $K_n^r$

There are sub-semi-groups that should be singled out:

1) $\Sigma_n^r$: the semi-group of spherical knots ;
2) $S_n^r$: the semi-group of locally unknotted spherical knots ;
3) $H_n^r$: the semi-group of homogeneous spherical knots (See [2]).

## § 6. General Position and Orthotopy — Part I.

Although our ultimate concern will be with isotopies, we shall have to deal with something not quite as restrictive in search of isotopy.

DEFINITION 3. *A local isotopy* $\varphi_t : K \to E^r$, will be a map $\varphi : I \times K \to E^r$ which is simplicial for a fixed subdivision of K and for each $t$. It is nonsingular on each simplex of K, for each $t$, and piecewise linear in $t$ for fixed $p$, the subdivision of I being independent of $p$.

DEFINITION 4. *An orthomorphism* $\varphi : K \to E^r$ is a simplicial map, nonsingular on each simplex in K, and satisfies the following condition (which assures that self-intersections of K are not too high in dimension):

If $\Delta_1$, $\Delta_2$ are distinct simplices in K such that $\varphi(\text{int } \Delta_1) \cap \varphi(\text{int } \Delta_2)$ is non-empty, then,

$$\text{codim } R(\Delta_\alpha, \Delta_\beta) \leq 1.$$

DEFINITION 5. An *orthotopy* $\varphi_t = K \to E^r$ is (i) an orthomorphism for each $t$ (ii) a local isotopy.

Essential to an analysis of the problem of knotted spheres in euclidean space is the following generalization of a theorem of Guggenheim.

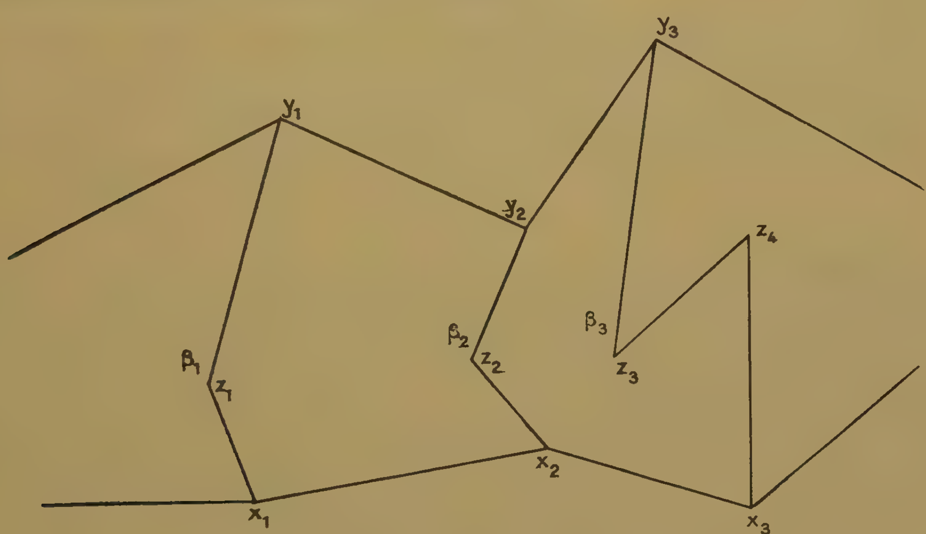


Fig. 1

THEOREM: Let K and K′ be simplicially isomorphic complexes $\psi : K \to K'$ imbedded piece-wise linearly in $E^r$. There is an orthotopy $\psi_t$ between K and K′. More precisely, there is an orthotopy $\psi_t : K \to E^r$ such that $\psi_0 = 1$ and $\psi_1 = \psi$.

Proof. Draw polygonal arcs $\beta_i$ from the vertices $w_i$ of K to the corresponding vertices $\psi(w_i)=w'_i$ of K'. See Fig. 1.

## § 7. Perturbation into General Position.

Let V be the set of all vertices of the $\beta_j$'s. Let $P_v$ for $v \in V$ stand for the set of all hyperplanes spanned by subsets of vertices in $V-\{v\}$.

Notice that $P_v$ is always a finite union of hyperplanes, hence a closed $(r-1)$-dimensional set.

Definition 6. I shall say: Figure 1 is in general position if $v \notin P_v$ for all $v \in V$. It will be a great simplification if the problem of proving the orthotopy theorem reduces to proving it for the case when Figure 1 is in general position.

This will be so if the following lemma is proven.

Lemma 2. It is possible to "put" the entire array $K \cup K' \cup (\cup_i \beta_i)$ of Figure 1 in general position by an arbitrarily slight isotopy.

Procedure: Order the vertices of V, $V=(v_1, \ldots, v_q)$. One can find a $v_1^{(1)}$ arbitrarily close to $v_1$, so that $v_1^{(1)} \notin P_{v_1}$. (For $P_{v_1}$ is of codimension one in $E^r$).

Lemma 3. There is an isotopy $\psi_t^{(1)}$ of the array of figure 1 which leaves all vertices other than $v_1$ fixed, and brings $v_1$ to a $v_1^{(1)}$ such that $v_1^{(1)} \notin P_{v'_1}$. In fact, $\psi^{(1)}$ is the identity on simplices outside of St $v_1$

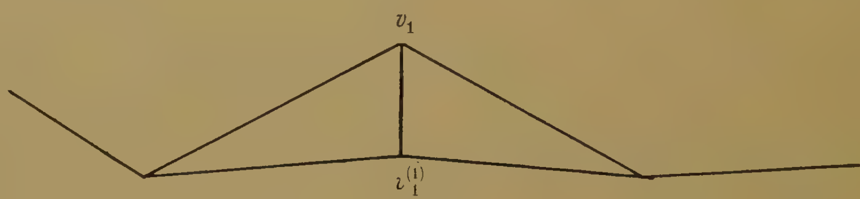


Fig. 2

and brings St $v_1 = J(v_1, \partial \mathrm{St}\, v_1)$ piecewise-linearly to $J(v_1^{(1)}, \partial \mathrm{St}\, v_1)$.

Now we study the new array, as perturbed by $\psi_1$. I will speak of $V^{(1)}$ as the new set of vertices $(V-\{v_1\}) \cup \{v_1^{(1)}\}$, and of $P_v^{(1)}$ as the union of hyperplanes generated by sets of points in $V^{(1)}-\{v\}$.

So, as matters nowst and we have $v_1^{(1)} \notin P^{(1)}_{v_1^{(1)}}$. The next stage in the process is similar. We must find a replacement $v_2^{(2)}$ for $v_2$ so close to $v_2$ that an isotopy $\psi_t^{(2)}$ can be found which leaves all vertices of the array other than $v_2$ fixed and sends $v_2$ linearly to $v_2^{(2)}$ and that $v_2^{(2)} \notin P^{(1)}_{v_2}$. But we need one more thing as well. We need $v_2^{(2)}$ to be taken so close to $v_2$ that the isotopy $\psi_t^{(2)}$ doesn't destroy the fact that $v_1^{(1)} \notin P^{(1)}_{v_1^{(1)}}$, since $P^{(1)}_{v_1^{(1)}}$ changes under the isotopy $\psi_t^{(2)}$. But it is clear that it can be so arranged. Thus we obtain a new array, $P^{(2)}_{v_i^{(2)}}$, and repeat the process.

And so it goes. At the $i^{th}$ stage, it is a question of isotopically perturbing $v_i^{(i-1)}$ to $v_i^{(i)}$ where $v_i^{(i)} \notin P^{(i)}_{v_i^{(i)}}$, and so slightly that one's previous handiwork:

$$v_k^{(i)} \notin P^{(i)}_{v_k^{(i)}} \qquad i > k$$

remains intact. The procedure ends with its final array in general position, proving the lemma.

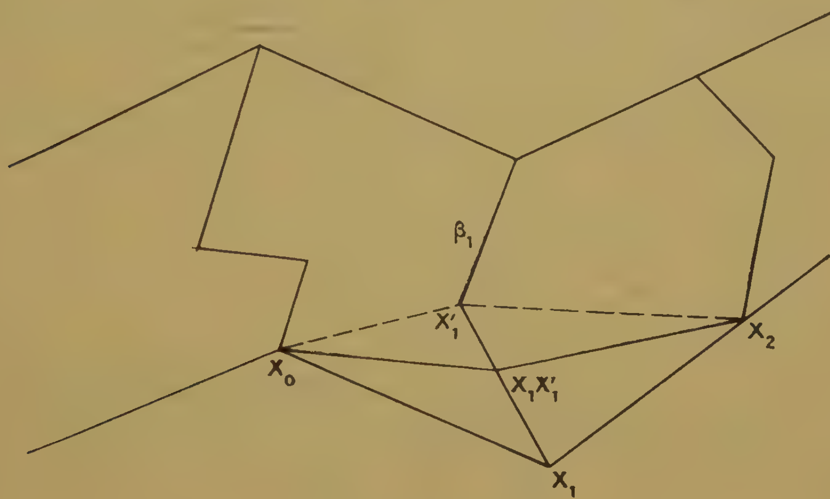


Fig. 3

The orthotopy $\varphi_t$ is obtained, step by step, climbing up the $\beta_j$'s. A typical step would consist in "replacing" one vertex, $x_1$ by the succeeding vertex, $x_1'$ on the arc $\beta_1$. In this manner, the orthotopy $\varphi_t$ will be obtained as the composite of a chain of orthotopies $\psi_t^{(i)}$, $i = 1, \ldots, \nu$, $\psi_t^{(i)}$ will be an orthotopy of the complex $K^{(i-1)}$ to $K^i$, where

$$K^0 = K$$
$$K^\nu = K'$$

and all $K^i$ will have as vertices only those in the array, $K^i$, being obtained from $K^{i-1}$ by chosing one vertex $x \in K^{i-1}$ and replacing the vertex $x$ by its successor $x'$ on the path of the array $\beta_x$, which contains $x$. This can be done, as long as $x$ is not the "last" vertex of $\beta_x$ ; or equivalently as long as $x \notin K'$. ("Successor" means in the direction towards $K'$ along $\beta_x$.) Thus the local isotopy $\psi_t^x$ which sends $x$ to $x'$ may be defined by its action on the vertices of $K^{i-1}$ (and extended piece-wise linearly to $K^{i-1}$):

$$\psi_t^x(v) = v \qquad \text{if } v \in V(K^{i-1})$$
$$\psi_t^x(x) = (1-t)x + ix' \qquad v \neq x$$

and $K^i$ is, of course, $\psi_1^x(K^{i-1})$. Since the number of vertices of the array is finite this process must terminate, if repeated enough, with a $K^\nu$ such that all vertices of $K^\nu$ are in $K'$, i.e. $K^\nu = K'$. Thus the chain of local isotopies $\psi_t^{(1)}, \ldots, \psi_t^{(\nu)}$ will yield an orthotopy $\varphi_t$ between $K$ and $K'$ if they themselves are orthotopies.

LEMMA 4. The $\psi_t^{(i)}$ are orthotopies.

Let $\psi_t = \psi_t^{(i)}$, dropping the unnecessary superscript. I shall prove:

LEMMA 5. For each $t$, $0 < t \leq 1$, $\psi_t$ is an orthomorphism.

Which clearly implies lemma 4 above, and by induction I assume $\psi_0$ to be an orthomorphism already.

Call $\Delta^t = \psi_t(\Delta)$ for $\Delta$ a simplex in $\widetilde{K} = K^{(i)}$. Assume that $\psi_t$ fails to be an orthomorphism for some $t > 0$. So: int $\Delta_1^t$, int $\Delta_2^t$ intersect, where $\Delta_i^t = \psi_t(\Delta_i)$ and $\Delta_1$, $\Delta_2$ are distinct simplices of K, yet:

$$\operatorname{cod}[R(\Delta_1^t, \Delta_2^t)] \geq 2$$

Let $x \in \widetilde{K}$ be the unique vertex moved by $\psi_t$, and, by our convention,

$$\psi_t(x) = x_t.$$

I must distinguish between two cases:

I) $$\Delta_1^t \in St(x_t), \quad \Delta_2^t \notin St(x_t)$$

II) $$\Delta_1^t, \ \Delta_2^t \in St(x_t).$$

CASE I: Let $\Delta_2^t = \Delta_2$ be the simplex unmoved by $\psi_t$. The assumption

$$\operatorname{cod}[R(\Delta_1^t, \Delta_2)] \geq 2$$

gives us

$$\operatorname{cod}[R(\Delta_1^t, \Delta_2, x_1)] \geq 1.$$

I make the notational convention: $\hat{\Delta}^t \subset \Delta^t$ is the face in $\Delta^t$ opposite the vertex $x_t$, for $\Delta_t \subset St(x_t)$. Thus:

*a)* $$\hat{\Delta}^t \subset \partial St(x_t)$$

*b)* $$\hat{\Delta}^t = \hat{\Delta}^0 \text{ for all } 0 \leq t \leq 1.$$

A useful fact for the arguments that follow is the obvious:

LEMMA 6. Let S be a set, $S \subset E^r$, $x, y \in E^r$ and $a \in R$, $a \neq 1$, then:

$$ax + (1-a)y \in R(S), \ x \in R(S)$$

implies $y \in R(S)$.

A) Assuming (I), then $t \neq 1$.

PROOF: If $t = 1$, then

$$R(\hat{\Delta}_1^t, \Delta_2) \ni x_1,$$

for let $\alpha_1 \in$ int $\Delta_1^1$, $\alpha_2 \in$ int $\Delta_2$ and $\alpha_1 = \alpha_2 = \lambda\xi_1 + (1-\lambda)x_1$, for $\xi_1 \in \hat{\Delta}_1^1$, and $0 < \lambda < 1$. Thus $\alpha_2 = \lambda\xi_1 + (1-\lambda)x_1 \in R(\hat{\Delta}_1^1, \Delta_2)$ and $\xi_1 \in R(\hat{\Delta}_1^1, \Delta_2)$ but, by Lemma 6, one has $x_1 \in R(\hat{\Delta}_1^1, \Delta_2)$; however

$$\operatorname{cod} R(\hat{\Delta}_1^1, \Delta_2) \geqslant \operatorname{cod} R(\Delta_1^1, \Delta_2) \geq 2$$

therefore

$$x_1 \in R(\hat{\Delta}_1^t, \Delta_2) \subset P_{x_1},$$

which contradicts general positionality. Therefore $0 < t < 1$.

B) $R(\Delta_1^0, \Delta_2) \subset R(\Delta_1^t, \Delta_2, x_1)$.

To demonstrate this, it suffices to show

$$x_0 \in R(\Delta_1^t, \Delta_2, x_1).$$

But $x_1, x_t \in R(\Delta_1^t, \Delta_2, x_1)$ and since $x_t = tx_1 + (1-t)x_0$ and $t \neq 1$, Lemma 6 again gives

$$x_0 \in R(\Delta_1^t, \Delta_2, x_1).$$

Also, $x_1 \in R(\Delta_1^0, \Delta_2)$: Because if $\alpha_1 \in \operatorname{int} \Delta_1^t, \alpha_2 \in \operatorname{int} \Delta_2, \alpha_1 = \alpha_2$, then $\alpha_2 = \alpha_1 = \lambda\xi_1 + (1-\lambda)x_1$, $\xi_1 \in \Delta_1^0$ and $0 < \lambda < 1$, but

$$\begin{aligned} \operatorname{cod} R(\Delta_1^0, \Delta_2) &\geq \operatorname{cod} R(\Delta_1^t, \Delta_2, x_1) \\ &\geq \operatorname{cod} R(\Delta_1^t, \Delta_2) - 1 \geq 1. \end{aligned}$$

Therefore

$$x_1 \in R(\Delta_1^0, \Delta_2) \subset P_{x_1}$$

again contradicting general positionality.

Case II: Assume again that $\psi_t$ is not an orthomorphism for some $t > 0$.

There are simplices $\Delta_1^t, \Delta_2^t$ such that:

1) $$\alpha^t \in \operatorname{int} \Delta_1^t \cap \operatorname{int} \Delta_2^t$$

2) $$\operatorname{cod} R(\Delta_1^t, \Delta_2^t) \geq 2.$$

A) $$R(\Delta_1^0, \Delta_2^0) \subseteq R(\Delta_1^t, \Delta_2^t, x_0),$$

an evident fact, implying

$$\operatorname{cod} R(\Delta_1^0, \Delta_2^0) \geq \operatorname{cod} R(\Delta_1^t, \Delta_2^t, x_0) \geq 1.$$

B) In fact:

$$\operatorname{cod} R(\Delta_1^0, \Delta_2^0) \geq 2.$$

For, if $\operatorname{cod} R(\Delta_1^0, \Delta_2^0) = 1$,

$$R(\Delta_1^0, \Delta_2^0) = R(\Delta_1^t, \Delta_2^t, x_0)$$

and $x_1 \in R(\Delta_1^t, \Delta_2^t, x_1)$.

Since $x_t, x_0 \in R(\Delta_1^t, \Delta_2^t, x_0)$ and $x_t = (1-t)x_0 + tx_1$, $t \neq 0$, this implies:

$$x_1 \in R(\Delta_1^0, \Delta_2^0) \subset P_{x_1}$$

contradicting general positionality.

Let $\alpha_i^0 \in \operatorname{int} \Delta_i^0$ be the elements for which $\psi_t(\alpha_i) = \alpha^t$.

C) $\alpha_1^0 \neq \alpha_2^0$. For, by (B), $\operatorname{cod} R(\Delta_1^0, \Delta_2^0) \geq 2$, and $\psi_0$ being an orthomorphism, $\operatorname{int} \Delta_1 \cap \operatorname{int} \Delta_2$ is empty. Let $\alpha_i^0 = \delta_i^0 + \lambda_i x$, where

$$\left(\frac{1}{1-\lambda_i}\right) \delta_i^0 \in \hat{\Delta}_i^0.$$

Then:

$$\psi_t(\alpha_i) = \alpha^t = \delta_i^0 + \lambda_i x^t$$
$$= \delta_i^0 + \lambda_i[(1-t)x_0 + tx_1]$$

giving us

D) $$\delta_1^0 - \delta_2^0 = (\lambda_2 - \lambda_1)[(1-t)x_0 + tx_1].$$

Also:

E) $$0 \neq \alpha_1^0 - \alpha_2^0 = (\delta_1 - \delta_2) + (\lambda_1 - \lambda_2)x_0.$$

F) $$\lambda_1 - \lambda_2 \neq 0.$$

If $\lambda_1 = \lambda_2$, one would have, by D),

$$\delta_1^0 = \delta_2^0, \ \alpha_1^0 = \alpha_2^0$$

which would contradict E). So F) follows.

G) $$\frac{\delta_1^0 - \delta_2^0}{\lambda_2 - \lambda_1} \in R(\Delta_1, \Delta_2)$$

for $$\frac{\delta_i^0}{1-\lambda_i} \in \hat{\Delta}_i^0 \subset R(\Delta_1, \Delta_2).$$

H) $x_1 \in R(\Delta_1, \Delta_2)$, for D) and G) yield

$$\frac{\delta_1^0 - \delta_2^0}{\lambda_2 - \lambda_1} = (1-t)x_0 + tx_1 \in R(\Delta_1, \Delta_2).$$

clearly $x_0 \in R(\Delta_1, \Delta_2)$, and by the induction assumption, $t \neq 0$; H) follows by the application of Lemma 6. But H) contradicts general positionality, since

$$x_1 \in R(\Delta_1, \Delta_2) \subset P_{x_1}.$$

So the orthotopy theorem is proved. With just a bit more care in the proof of the theorem, we could have proved this slightly strengthened version which will be needed later.

THEOREM (EXTENSION). Let $F_0$, $F_1$ be imbeddings (or merely orthomorphisms, for that matter) of K in $E^r$. Let $L \subset K$ be a subcomplex and

$$f_t : L \to E^r$$

an orthotopy such that

$$f_0 = F_0 | L, \ f_1 = F_1 | L.$$

Then there is an orthotopy $F_t$ between $F_0$ and $F_1$ such that

$$F_t | L = f_t.$$

## § 8. The Singularity Locus.

DEFINITION 7. The pre-locus V of an orthomorphism $f : K \to E^r$ is the set of multiple points of $f$ in K. That is,

$$V = \{k \in K \mid \exists k' \neq k, f(k') = f(k)\}.$$

Clearly V is a subcomplex of K. The *locus* L is the image of the pre-locus in $E^r$,

$$L = f(V).$$

The *pre-locus* (and *locus*) of an orthotopy $f_t$, $0 \leq t \leq 1$, is the union of all pre-loci $V_t$ (loci) of the orthomorphisms $f_t$ for each $t$, $0 \leq t \leq 1$:

$$V = \bigcup_{t \in I} V_t.$$

And again, V is a subcomplex of K.

Lemma 7. Let $f: K^n \to E^r$, where $K^n$ is an $n$-complex, be an orthomorphism, and V its singularity pre-locus. Then

$$\dim V \leq 2n - r + 1.$$

If $f_t: K^n \to E^r$ is an orthotopy, and W its pre-locus, then

$$\dim W \leq 2n - r + 2.$$

Proof: Let $p \in V$. Then $p \in \Delta_1$, $p \in \Delta_2$, where $\Delta_i$ are the images of distinct simplices of K under $f$.

$$\dim \Delta_1 \cap \Delta_2 \leq \dim R(\Delta_1) \cap R(\Delta_2) = \dim R(\Delta_1) + \dim R(\Delta_2) - \dim R(\Delta_1, \Delta_2).$$

But if $\Delta_1$, $\Delta_2$ have an interior intersection at all, $\dim R(\Delta_1, \Delta_2) \geq r - 1$. So

$$\dim \Delta_1 \cap \Delta_2 \leq \dim R(\Delta_1) + \dim R(\Delta_2) - (r - 1)$$

and since $\dim R(\Delta_i) \leq n$

$$\dim \Delta_1 \cap \Delta_2 \leq 2n - r + 1.$$

Corollary 1 (Guggenheim). Any two imbeddings $\varphi_0$, $\varphi_1: K^n \to E^r$ are isotopic if $r \geq 2n + 2$.

For by the orthotopy theorem, there is an orthotopy $\varphi_t$ between $\varphi_0$ and $\varphi_1$. And by the above the dimension of the singularity locus of each orthomorphism $\varphi_t$ is $-1$, or the singularity locus of each $\varphi_t$ is empty. Therefore, the orthotopy is an isotopy.

Corollary 2. Let L′, L, N be subcomplexes of $E^r$ and $\varphi: L \to L'$ an isomorphism leaving $L \cap N$ fixed. Then there is an orthotopy $\varphi_t$ from L to L′, leaving $L \cap N$ fixed ($\varphi_1 = \varphi$), such that if $\varphi_t(\Delta)$ and $\Delta'$ have a non-empty interior intersection, for $\Delta \in L - L \cap N$, and $\Delta' \in N$, then $\operatorname{cod} R(\varphi_t(\Delta), \Delta') \geq 1$.

Proof: Apply the orthotopy theorem with $K = L \cup N$, $K' = L' \cup N$.

Corollary 3. In the situation of the above corollary, if

$$r \geq \dim L + \dim N + 2$$

the orthotopy $\varphi_t$ can be chosen to be an isotopy of the complex $L \cup N$. Thus $\varphi_t(l) \in N$ implies $l \in N$ for $l \in L$.

This corollary is interpreted as saying that L and N are unlinkable in $E^r$ if

$$r \geq \dim L + \dim N + 2.$$

## § 9. Some Necessary Facts Concerning General Positionality.

1) *Stability of Orthotopy.*

LEMMA 8. Let $\varphi_t : K \to E^r$ be an orthotopy; then there is a number $\rho(\varphi_t) > 0$ such that if $\varphi'_t : K \to E^r$ is a continuous family of simplicial maps, such that

$$||\varphi_t(v) - \varphi'_t(v)|| < \rho(\varphi_t)$$

for all $t$, and all vertices $v \in V(K)$, then $\varphi'_t$ is again an orthotopy.

The proof is a rewording of the proof of Lemma 1 of [3]. I omit it.

2) Even stronger than an orthomorphism is a map $f : K \to E^r$ such that: 1) $f$ is a simplicial map non-singular on each simplex of K; 2) if int $\Delta_1$ and int $\Delta_2$ intersect, for $\Delta_1$, $\Delta_2$ distinct simplices of K, then $R(\Delta_1, \Delta_2) = E^r$. Just to give such an $f$ a name, I call it *maximally transverse.*

LEMMA 9. If $f : K \to E^r$ is a simplicial map, it is approximable arbitrarily closely by a maximally transverse map

$$f' : K \to E^r.$$

Moreover, if $L \subset K$ and $f|L$ is already maximally transverse, one can have $f'|L = f|L$.

The method of proof has been displayed sufficiently often that I omit the precise proof (or statement) of this lemma.

COROLLARY 4. Any map $f : K^n \to E^r$ for $r \geq 2n + 1$ may be approximated arbitrarily closely by an imbedding $f' : K \to E^r$.

COROLLARY 5. Let $A, B \subset C$ be subcomplexes and virt $\dim_B A +$ virt $\dim_A B + 1 \leq r$. Let

$$\varphi_1 : A \to E^r$$
$$\varphi_2 : B \to E^r$$

be simplicial maps such that $\varphi_1 | A \cap B = \varphi_2 | A \cap B$. Then there is a simplicial map $\varphi : A \cup B \to E^r$ such that $\varphi | A = \varphi_1$, $\varphi | B$ approximates $\varphi_2$, and $\varphi(B - B \cap A)$ is disjoint from $\varphi(A)$. If the $\varphi_i$ were homeomorphisms then so will $\varphi$ be.

Define the map $\varphi' : A \cup B \to E^r$ to be the composite of $\varphi_1$ on A and $\varphi_2$ on B. Then $\varphi' | A$ is already maximally transverse. Approximate $\varphi'$ by $\varphi$, a maximally transverse map $\varphi : A \cup B \to E^r$, such that $\varphi | A = \varphi' | A$, and so close to $\varphi'$ so that $\varphi | B$ is still an imbedding of B (applying the Stability Lemma for imbeddings, Lemma 1 of [3]). $\varphi$ will be an imbedding of $A \cup B$ if

$$\dim A + \dim B + 1 \leq r$$

and, after a simple modification (which I omit) it would be an imbedding even if

$$\text{virt} \dim_B A + \text{virt} \dim_A B + 1 \leq r.$$

## § 10. Part II : The Main Theorem.

I shall use the tools developed in Part I to prove the following theorem: Let S be a $k$-sphere knot in $E^r$ which is homogeneous. Then if

$$r \geq \frac{3k+5}{2}$$

S is invertible. Or, in terms of the semi-groups of [3] in the same range of dimensions, as above

$$H^r_k = G^r_k$$

Coupled with the main theorem in [1], one has: In the same range of dimensions, all homogeneous knots are $*$-trivial. Indeed, with no further complication, let $f_t$ be an orthotopy between two manifolds K and K′ in $E^r$ satisfying assumption (o):

(o) 1) $r \geq \dfrac{3k+5}{2}$.

2) K is homogeneous.

3) The singularity locus $V \subset K$ of $f_t$ can be brought into a $k$-cell $\Delta^k \subseteq K$ by a continuous family of homeomorphisms $h_t : K \to K$ such that $h_0 = 1$, and $h_1 : V \to \Delta^k$. Clearly condition 3) follows if K is a sphere.

THEOREM 2. If $f_t$ is an orthotopy between K and K′ satisfying condition (o), then :

$$K' = K + S$$

where S is a spherical knot.

The paragraph titles together with the accompanying diagrams provide a rough outline of the method of proof of the theorem.

## § 11. Isolation of the Singularity Locus.

Let $f_t$ be an orthotopy of K′ to K with singularity pre-locus $V \subset K$. Thus

$$f_t : K \to E^r,$$

$f_1 : K \to K \subset E^r$ is the natural injection, and $f_0 : K \to K' \subset E^r$. I should like to find a neighborhood $U_1$ of $f(I \times V) \subset E^r$ for which there exists a regular neighborhood N of V in K, such that

$$f_t : \partial N \to \partial U$$
$$f_t : N \to U.$$

Then U would serve to isolate that part of the orthotopy which had singularities. This would allow us to redefine $f_t$ on U so that the newly-defined $f^*_t$ would have no singularity on U. The resulting imbedding $K^* = f^*_1(K)$ would be equivalent to K′ and "differ from" K merely in U, a set of low virtual dimension,

$$\text{virt dim } U \leq \dim (V \times I).$$

The proof of the main theorem would then follow fairly easily.

## § 12. Regularizing the Orthotopy.

In order to carry out this program one must first replace the orthotopy $f$ by a close approximation $f'$, which has the property that $f'(I \times N)$ is disjoint from $f'(I \times (K-N))$ for N some regular neighborhood of the singularity prelocus V.

LEMMA 10. There is an orthotopy $f' : I \times K \to E^r$ arbitrarily close to $f$ which still has pre-locus V, and has the property that: $f'(I \times V)$ is disjoint from $f'(I \times (K-V))$.

PROOF: Apply corollary 5 of section 9 where $A = f'(I \times N)$, $B = f'(I \times K - N))$ in the notation of the corollary. One must check that

$$\text{virt dim}_B f'(I \times N) + \text{virt dim } B + 1 \leq r$$

Or:

$$2k - r + 3 + k + 1 \leq r.$$

But

$$\frac{3k+4}{2} \leq r$$

which proves the lemma.

1) *Isolation Lemma:* There is:

(1) A one-parameter family $U_s$, $0 \leq s \leq 1$, of closed neighborhoods of $f(V \times I)$ in $E^r$, and a continuous family of simplicial homomorphisms $g_s : U_i \to U_s$.

2) A one-parameter family $N_s$, $0 \leq s \leq 1$ of closed neighborhoods of V in $K'$, and a continuous family $p_s : N_1 \to N_s$, of simplicial homeomorphisms

— such that:

3) The map $g : \partial U_1 \times I \to E^r$ is a homeomorphism, where $g$ is

$$g(u, t) = q_t(u), \quad u \in \partial U_1.$$

4) The map $g : \partial N_1 \times I \to K$ is a homeomorphism, where $g$ is

$$g(n, t) = p_t(n) \quad n \in \partial N_1.$$

5)
$$f_t : N_s \to U_s$$
$$f_t : \partial N_s \to \partial U_s$$

and

6) The following diagram is commutative:

$$\begin{array}{ccccc} \partial N_1 \times I & \xrightarrow{g} & N_1 - \text{int } N_0 & \subset & K \\ \downarrow{\scriptstyle F_t} & & \downarrow{\scriptstyle f_t} & & \downarrow{\scriptstyle f_t} \\ \partial U_1 \times I & \xrightarrow[g]{} & U_1 - \text{int } U_0 & \subset & E^r \end{array}$$

where

$$F_t(n, t) = (f_t(n), t) \quad n \in \partial N_1$$

7) $\partial U_s$ is a homogeneous manifold combinatorially imbedded in $E^r$, $0 \leq s \leq 1$.

8) $$U_1 \cap f(I \times K) = f(I \times N_1).$$

PROOF: It is standard that one can choose a one-parameter family of regular closed neighborhoods of V in K′ with the properties that:

1) There is a continuous family $p_s : N_1 \to N_s$ of combinatorial homeomorphisms.

2) The map $g : \partial N_1 \times I \to K'$ is a simplicial homeomorphism, where $g$ is $g(n, t) = p_t(n)$ $n \in \partial N_1$.

Moreover, after Lemma 10, I assume $f$ to be such that $f(I \times N_1)$ is disjoint from $f(I \times (K - N_1))$. It then follows that $f(I \times N_s)$ is disjoint from $f(I \times (K - N_s))$. Now let $M_s = f(I \times N_s) \subset E^r$, and choose a combinatorial (continuous), monotonic increasing function $\varepsilon(s) > 0$, so small that $R_{\varepsilon(s)}(V)$ is disjoint from $\partial N_s$.

## § 13. Explicit Description of $U_s$.

Let

$$d(p) = d(p, f(V \times I))$$
$$d_s(p) = d(p, f[(K - \text{int } N_s) \times I])$$

for $p \in E^r$.

Define

$$U_s(p) = B_{\lambda_s(p)}(p)$$

where

$$\lambda_s(p) = \min\left[\left(\frac{d_s(p)}{d(p) + d_s(p)}\right) . \varepsilon(s), d_s(p)\right]$$

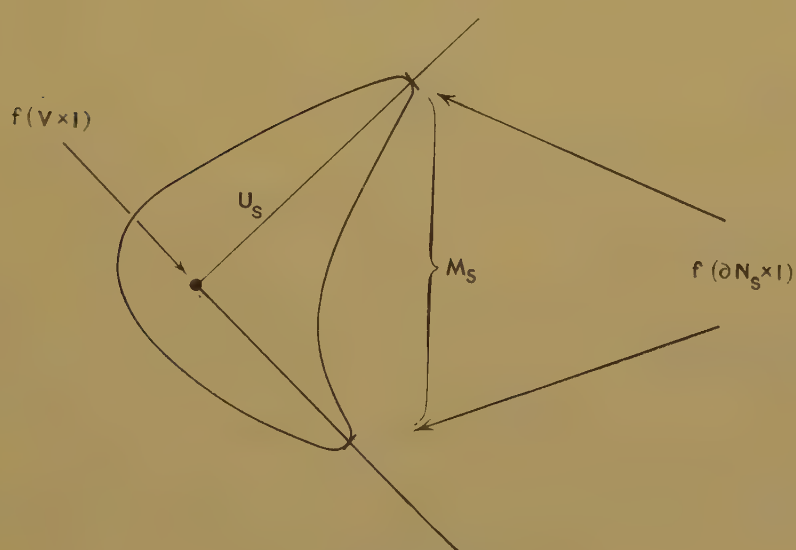


Fig. 4

and

$$U_s = \bigcup_{p \in M_s} U_s(p).$$

## § 14. Pictorial description of $U_s$.

I represent, in figure 4, $M_s$ by the V-shaped arc ; the vertex represents $f(V \times I)$ and the endpoints represent $f(\partial N_s \times I)$. Then $U_s$ is obtained simply by "thickening" every point on $f(\text{int } N_s \times I)$ a very little bit, the amount of thickening decreasing to zero as one approaches $f(\partial N_s \times I)$. The closure of this is $U_s$.

The proof that $U_s$, so defined, actually satisfies properties (1) thru (8) is straightforward ; I omit it therefore.

## § 15. $K^*$: The Modification of K.

Lemma 11. $f_t | \partial N_1$ is an isotopy of $\partial N_1$ in $\partial U_1$.

Proof: For $f_t$ is an orthotopy and the singularity pre-locus V is disjoint from $\partial N_1$.

Thus let $F_t : \partial U_1 \to \partial U_1$ be an ambient homeomorphism covering $f_t$, applying Theorem 2 of [3].

Lemma 12. There is a continuous family of homeomorphisms $G_t : \partial U_1 \times I \to \partial U_1 \times I$ such that

$$G_t(u, 1) = F_t(u)$$
$$G_t(u, 0) = u$$

for $u \in U_1$.

Proof: Define $G_t(u, s) = F_{st}(u)$ for $u \in U_1$, $0 \leq t, s \leq 1$ ; now define a homeomorphism

$$G_t^{(1)} : U_1 \to U_1$$

by

$$G_t^{(1)}(u) = u, \quad \text{if} \quad u \in U_0$$
$$G_t^{(1)}(u) = g G_t g^{-1}(u), \quad \text{if} \quad u \in U_1 - U_0$$

where $g$ is the homeomorphism

$$g : \partial U_1 \times I \to U_1 - \text{int } U_0$$

of the isolation lemma.

Notice that:

$$G_1^{(1)} f_0(n) = f_1(n) \quad \text{if} \quad n \in \partial N_1.$$

Define $h : K' \to E^r$ to be the composite

$$h(x) = f_1(x), \ x \in K' - \text{int } N_1,$$
$$h(x) = G_1^{(1)} f_0(x), \ x \in N_1.$$

The two definitions agree on $\partial N_1$, and $h$ is actually a homeomorphism since $f_1(K'—N_1)$ is disjoint from $U_1$. The image:

$$K^* = h(K')$$

is the necessary modification.

LEMMA 13. $K^* \sim K'$.

PROOF: One must obtain a homeomorphism $H: E^r \to E^r$ carrying $K'$ to $h(K')$. $G_1^{(1)}$ does this on $U_1$. In the bounded manifold $M = E^r—\text{int } U_1$, $f_t$ is an isotopy of $K'—\text{int } N_1 = L$ with the property that $f_t(\partial L) \subseteq \partial M$, and $f_t|\partial L$ is covered by an ambient isotopy $G_t^{(1)}|\partial M = \partial U_1$. Using Theorem 2 of [3] again, one can find an ambient isotopy $H_t$ covering both $G_t^{(1)}|\partial M$ and $f_t|\partial L$. Then the homeomorphism

$$H(x) = H_1(x) \qquad x \in E^r—\text{int } U_1$$
$$H(x) = G_1^{(1)}(x) \qquad x \in U_1$$

sends $K'$ to $K^*$, establishing their equivalence.

## § 16. Summarizing the Relevant Properties.

1) $K^* \sim K'$ ;
2) $K \cap U_1 = f_1(N_1)$ ;
3) $K^* \subset K \cup U_1$ ;
4) $K \cap E^r_-$ consists of a $k$-cell, $E^k_-$, imbedded as the standardly imbedded lower hemisphere of $S^k$ ;
5) virt $\dim_k U_1 \leq \dim(V \times I)$.

## § 17. Bringing $U \cap K$ into the Lower Half-Plane.

LEMMA 14. There is a homeomorphism

$$P: E^r \to E^r$$

which has the properties:

1) $P: K \to K$ ;
2) $P: K \cap U \to E^r_-$, the lower half-plane.

For since $V \subset K$, the singularity locus, is assumed contractible to $E^k_- \subset K$ ((3) of condition (0)), there is a continuous family $p_t: K \to K$, such that $p_0 = 1$ and $p_1: V \to E^k_-$. Since N is a regular neighborhood of V, a continuous family $p_t$ can be found which also has the property:

$$p_1: N \to E^k_-.$$

By homogeneity of K, $p_1$ can be extended to a homeomorphism $P: E^r \to E^r$.

Let

$$P(K^*) = K_2^*$$

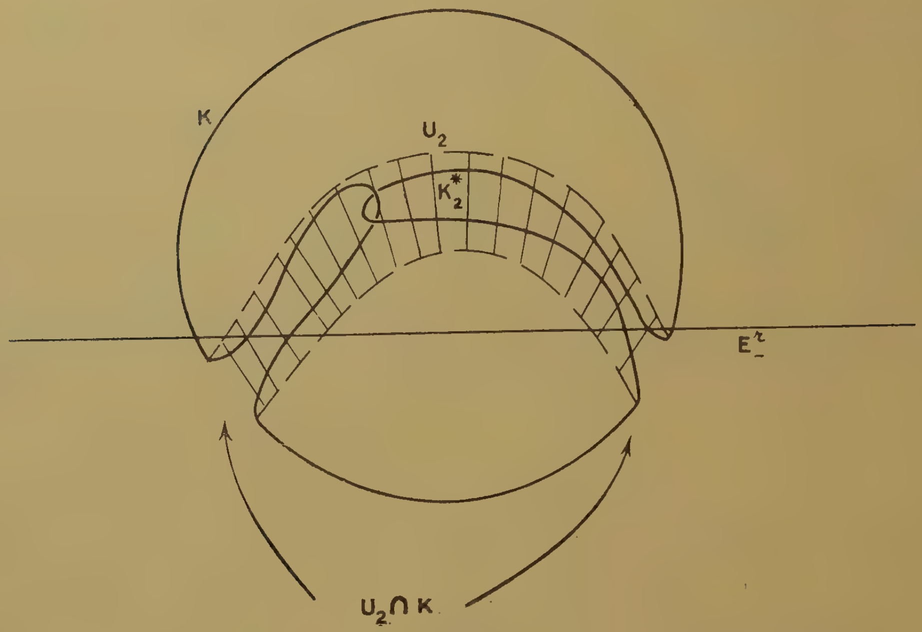


Fig. 5

and

$$P(K) = K$$
$$P(U_1) = U_2.$$

Then one has:

1) $U_2 \cap K_2'$ is in the lower half-plane;
2) $K_2^* \sim K_2'$;
3) $K_2^* \subset K \cup U_2$;
4) virt $\dim_K(U_2) \leq \dim(V \times I)$.

## § 18. Bringing $U_2$ into the Lower Half-Plane.

LEMMA 15. There is a homeomorphism

$$f_1 : U_2 \rightarrow E_-^r.$$

leaving $U_2 \cap K$ fixed.

PROOF: Obvious.

LEMMA 16. There is a homeomorphism $f_2 : U_2 \rightarrow E_-^r$ such that

1) $U_2 \cap K$ is left fixed;
2) $f_2(U_2 - U_2 \cap K) \subset E^r - K \cap E_-^r$.

PROOF: It is a simple application of Corollary 5, after one checks that

$$\text{virt} \dim_K U_2 + \text{virt} \dim K + 1 \leq r$$

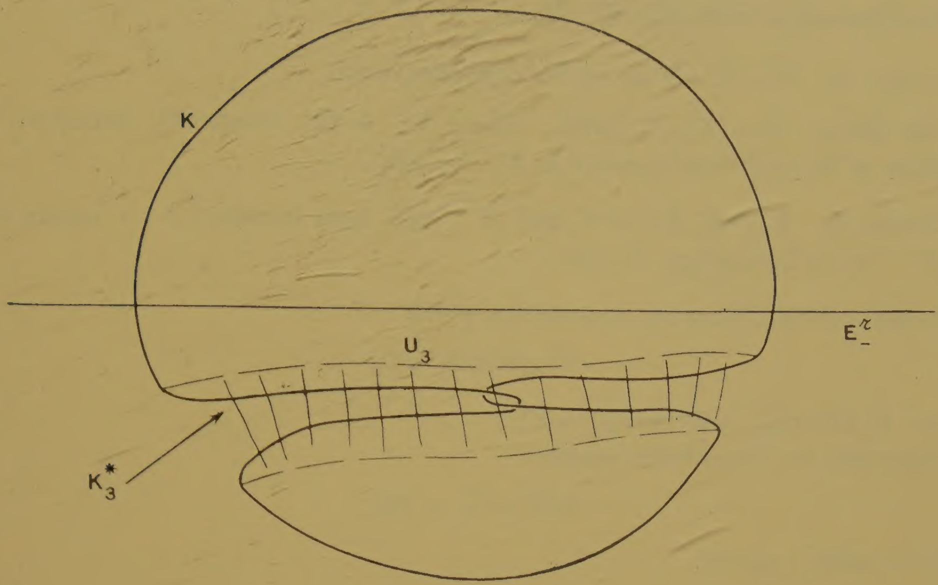


Fig. 6

or

$$2k - r + 3 + k + 1 \leq r$$

But this is the case, for

$$\frac{3k+4}{2} \leq r.$$

LEMMA 17. There is an isotopy $k_t: E^r \to E^r$ such that

1) $k_t | K_2 = 1$ ;

2) $k_1 | U_2 = f_2$ (thus $k_1(U_2) \subset E^r_-$).

PROOF: Apply Corollary 3 with $U_2 = L$, $f_2(U_2) = L'$, $K = N$ and observe that

$$\text{virt dim}_K U_2 + \text{virt dim } K + 2 \leq r$$

or

$$2k - r + 3 + k + 2 \leq r$$

for

$$\frac{3k+5}{2} \leq r.$$

Call:

$$k_1(K_2^*) = K_3^*, \quad k_1(U_2) = U_3$$

and: 1) $K_3^* \sim K_2^* \sim K'$

2) $K_3^* \subset K \cup U_3$

3) $U_3 \subset E^r_-$;

therefore:

$$K_3^* \cap E^r_+ = K \cap E^r_+.$$

## § 19. The Decomposition : $K \sim K' + S$.

LEMMA 18. $K_3^* = K + S$ where S is a spherical knot.

For define: $S = (K_3^* \cap E_-^r) \cup E_+^k$, where $E_+^k$ is the standardly imbedded upper hemisphere of the standard $k$-sphere in $E^r$, $E_+^k \subset E_+^r$.

LEMMA 19. $E_-^r \cap K_3^*$ is a $k$-cell, and $E^{r-1} \cap K_3^*$ is the standard $k-1$ sphere in $E^{r-1}$, where $E^{r-1}$ is the hyperplane $E_-^r \cap E_+^r$.

PROOF: Obvious from the construction of $K_3^*$:

$$E_-^r \cap K_3^* \subset E_-^k \cup U_3$$

(Because $K_3^* \subset K \cup U_3$, and $K \cap E_-^r = E_-^k$.)
Therefore the boundaries match:

$$\partial(E_-^r \cap K_3^*) = \partial E_+^k$$

and S is actually a sphere.

LEMMA 20. $K + S = K_3^*$.

This is obvious. ($S \cap E_+^r$ is the standard $k$-cell, $E_+^k$, $K \cap E_-^r$ is the standard $k$-cell $E_-^k$. Therefore their sum consists simply of:

$$(K - \text{int } E_-^k) \cup (S - \text{int } E_+^k) = X.$$

But this X is just $K_3^*$.

REFERENCES


[1] GUGGENHEIM, Piece-wise Linear Isotopy, *Journal of the London Math. Soc.*, vol. 46.
[2] B. MAZUR, Certain Semi-Groups of Spherical Knots (this *Journal*).
[3] B. MAZUR, On the definition of equivalence of combinatorial imbeddings (this *Journal*).


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